

Involutive Varieties with Smooth Support

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We prove a structure theorem for conic involutive varieties of the cotangent bundle of a smooth algebraic variety X whose projection on X is smooth. The paper includes an example of a conic involutive variety of T^*X which is not the characteristic variety of any \mathcal{D}_X -module. © 1997 Academic Press

1. INTRODUCTION

The most important geometric invariant in the theory of \mathcal{D} -modules is the characteristic variety. Let X be a smooth complex algebraic variety and denote by \mathcal{D}_X its sheaf of rings of differential operators. If \mathcal{M} is a coherent sheaf of modules over \mathcal{D}_X then its characteristic variety $\text{Ch}(\mathcal{M})$ is a subvariety of the cotangent bundle T^*X . The commutator of operators in \mathcal{D}_X is closely related to the Poisson bracket induced by the symplectic form of T^*X . Using this one can show that $\text{Ch}(\mathcal{M})$ is an involutive subvariety of T^*X . Geometrically this means that the skew-orthogonal complement of the tangent space of $\text{Ch}(\mathcal{M})$ at each regular point is contained in the tangent space itself.

The fact that characteristic varieties are involutive severely restricts them. For example, an involutive variety of T^*X cannot have dimension less than that of X . In this paper we discuss another one of these

restrictions. Denote by π the projection of T^*X on the base X . The main result of this paper (Corollary 3.2) is a characterization of those involutive conical subvarieties W of T^*X whose projection $\pi(W)$ is smooth in X . This can be seen as a geometric analogue of a famous theorem of Kashiwara (Theorem 4.1) on the structure of coherent \mathcal{D}_X -modules whose support in X is smooth.

Combining our result with Kashiwara's theorem we prove in Sect. 4 that certain involutive varieties of T^*X are characteristic varieties of \mathcal{D}_X -modules. Our results offer a very partial answer to the question: Which conical involutive varieties of T^*X are characteristic varieties of \mathcal{D}_X -modules?

The proof of the main theorem is split between Sects. 2 and 3. In particular the "local case" is dealt with in Sect. 2 in greater generality than is actually needed in Sect. 3. Several applications are considered in Sect. 4. Finally, in Sect. 5 we give an example of a conical involutive variety of T^*X which is not the characteristic variety of a \mathcal{D}_X -module.

2. THE LOCAL CASE

We begin with a result from commutative algebra. Let K be a field of characteristic zero and let A be a commutative K -algebra. We will denote the module of K -derivations of A by $\text{Der}_K(A)$. Throughout this section we assume that A is an affine regular domain. Thus the ring of differential operators $\mathcal{D}(A)$ is generated by A and $\text{Der}_K(A)$.

The K -algebra $\mathcal{D}(A)$ has a filtration $\{\mathcal{D}^k(A)\}_{k \geq 0}$ by the order of a differential operator. Recall that $\mathcal{D}^0(A) = A$ and $\mathcal{D}^1(A) = A + \text{Der}_K(A)$. The graded algebra associated with this filtration is isomorphic to the symmetric algebra of $\text{Der}_K(A)$, which we denote by $S(A)$. The homogeneous component of degree k of $S(A)$ will be denoted by $S^k(A)$. The *symbol map* of order k ,

$$\sigma_k: \mathcal{D}^k(A) \rightarrow S^k(A) \cong \frac{\mathcal{D}^k(A)}{\mathcal{D}^{k-1}(A)}$$

is a K -linear map induced by the canonical projection.

Let $f_1, f_2 \in S(A)$ be homogeneous elements of degrees k_1 and k_2 , respectively. Choose $d_i \in \mathcal{D}(A)$ such that $\sigma_{k_i}(d_i) = f_i$, for $i = 1, 2$. The *Poisson bracket* of f_1 and f_2 is defined by

$$\{f_1, f_2\} = \sigma_{k_1+k_2-1}([d_1, d_2]). \quad (2.1)$$

The Poisson bracket can be extended by linearity to the whole of $S(A)$, thus making $S(A)$ into a Lie algebra. The *hamiltonian* of $f \in S(A)$ is the K -linear endomorphism h_f of $S(A)$ defined on $g \in S(A)$ by $h_f(g) = \{g, f\}$. It is a derivation of $S(A)$. An ideal I of $S(A)$ is *involutive* if $\{I, I\} \subseteq I$.

Now let $B = A/J$ be a homomorphic image of A . Since A is regular, we have that $\text{Der}_K(A, B) \cong B \otimes_A \text{Der}_K(A)$, as B -modules. Hence there exists an embedding of B -modules

$$\text{Der}_K(B) \rightarrow B \otimes_A \text{Der}_K(A), \tag{2.2}$$

see [6, Theorem 25.1]. Since the symmetric algebra construction is functorial, we obtain a homomorphism of B -algebras

$$S(B) \rightarrow B \otimes_A S(A).$$

Denote by ϕ the composition of this homomorphism with the isomorphism

$$B \otimes_A S(A) \cong \frac{S(A)}{S(A)J} = \overline{S(A)}.$$

Note that ϕ is not necessarily injective, even though the module homomorphism (2.2) is an embedding. If $f \in S(A)$, then \bar{f} will denote its image in $\overline{S(A)}$. A simple diagram chase shows that the image of $\text{Der}_K(B)$ under ϕ corresponds to the elements $\bar{\partial} \in \overline{S(A)}$, where $\partial \in \text{Der}_K(A)$ satisfies $\partial(J) \subseteq J$.

The following lemma will be used in the proof of the main theorem of this section. It is proved in [7, Sect. 14.6.4]. Recall that a derivation D of a K -algebra S is *locally nilpotent* if, given $a \in S$, there exists $k \in \mathbb{N}$ such that $D^k(a) = 0$.

2.3. LEMMA. *Let S be a K -algebra and let D be a locally nilpotent derivation of S . Suppose that for some $t \in S$ one has $D(t) = 1$. Then:*

- (1) $S = R[t]$, where R is the ring of constants of S .
- (2) t is algebraically independent over R .
- (3) $D = d/dt$.

2.4. THEOREM. *Let A be an affine regular domain over K . Let I be an involutive ideal of $S(A)$. Suppose that there exist $a \in I \cap A$ and $\partial \in \text{Der}_K A$ such that $\partial(a) = 1$ and $B = A/Aa$. Then \bar{I} is generated by $\phi(\phi^{-1}(\bar{I}))$ in $\overline{S(A)}$.*

Proof. Recall that $S(A) \cong \text{gr } \mathcal{L}(A)$. Hence we can identify a derivation (as an element of $S(A)$) with its symbol in $\text{gr } \mathcal{L}(A)$. Thus, let ξ be the symbol of ∂ . By (2.1),

$$\{\xi, a\} = \partial(a) = 1.$$

Since $[a, \mathcal{L}^k(A)] \subseteq \mathcal{L}^{k-1}(A)$, it follows that the hamiltonian h_a is a locally nilpotent derivation of $S(A)$. Note that $h_a(x) = 0$ if $x \in A$. Thus h_a gives

rise to a locally nilpotent derivation D on $\overline{S(A)} = S(A)/S(A)a$. If $\bar{\xi}$ denotes the image of ξ in $\overline{S(A)}$, then $D(\bar{\xi}) = 1$. By Lemma 2.3,

$$\overline{S(A)} \cong R[\bar{\xi}],$$

where R is the ring of constants of D . Clearly $\phi(B) \subseteq R$. As we have seen, an element of $\phi(\text{Der}_K(B))$ can be represented in the form $\bar{\eta}$, where η is the symbol of a derivation d of $\text{Der}_K(A)$, which satisfies $d(a) \in Aa$. Hence

$$D(\bar{\eta}) = \overline{\{\eta, a\}} = \overline{d(a)} = 0.$$

Thus $\phi(\text{Der}_K(B)) \subseteq R$. Since $S(B)$ is generated over B by $\text{Der}_K(B)$, we conclude that $\phi(S(B)) \subseteq R$.

We want to show that $R = \phi(S(B))$. Thus let d be a derivation of $\text{Der}_K(A)$, and denote its symbol by η . Then $(d - d(a)\partial)(a) = 0$. Hence $d - d(a)\partial$ induces a derivation on B . In particular $\overline{\eta - d(a)\xi} \in \phi(S(B))$. But,

$$\bar{\eta} = \overline{(\eta - d(a)\xi)} + \overline{d(a)\xi} \in \phi(S(B))[\bar{\xi}].$$

Hence $\overline{\text{Der}_K(A)} \in \phi(S(B))[\bar{\xi}]$. Since $S(A)$ is generated by $\text{Der}_K(A)$ over A , we conclude that $\overline{S(A)} = \phi(S(B))[\bar{\xi}]$, from which it follows that $R = \phi(S(B))$.

Now let $I' = \bar{I} \cap R$. Clearly $I'[\bar{\xi}] \subseteq \bar{I}$ and we will show that the two are equal. Since I is involutive and contains a , then \bar{I} satisfies

$$D(\bar{I}) \subseteq \bar{I}.$$

Let $\sum_{i=0}^k b_i \bar{\xi}^i \in \bar{I}$, where $b_0, \dots, b_k \in R$ and $b_k \neq 0$. Then $D^k(\sum_{i=0}^k b_i \bar{\xi}^i) = k!b_k \in \bar{I}$, and so $b_k \in \bar{I}$. By induction $b_k, \dots, b_0 \in \bar{I}$. Therefore $\bar{I} = I'[\bar{\xi}]$. Since $R = \phi(S(B))$, it follows that

$$\phi^{-1}(I') = \phi^{-1}(\bar{I} \cap R) = \phi^{-1}(\bar{I}).$$

Thus

$$\bar{I} = I'[\bar{\xi}] = \overline{S(A)}I' = \overline{S(A)}\phi(\phi^{-1}(\bar{I})),$$

as required. ■

Now assume that B is a quotient of A that is also regular. Then we have the embeddings

$$\mathcal{D}^*(B) \hookrightarrow B \otimes_A \mathcal{D}^*(A),$$

which in turn give embeddings

$$S^k(B) \cong \frac{\mathcal{D}^k(B)}{\mathcal{D}^{k-1}(B)} \hookrightarrow \frac{B \otimes \mathcal{D}^k(A)}{B \otimes \mathcal{D}^{k-1}(A)} \cong B \otimes_A S^k(A).$$

Thus the homomorphism of graded algebras $\phi: S(B) \rightarrow B \otimes_A S(A)$ is injective if B is regular.

Let f_1, f_2 be homogeneous elements of degrees k_1, k_2 in $S(A)$. Then $f_i = \sigma_{k_i}(d_i)$, with $d_i \in \mathcal{A}(A)$ and $i = 1, 2$. If \bar{f}_1, \bar{f}_2 belong to the image of ϕ , then we may compute their Poisson bracket in $S(B)$. Since ϕ is a graded embedding, this bracket is equal to the image of $[1 \otimes d_1, 1 \otimes d_2]$ in $B \otimes S^{k_1+k_2-1}(A)$. But this is $\overline{\{f_1, f_2\}}$. This implies that if I is an involutive ideal of $S(A)$, then $\phi^{-1}(\bar{I})$ is an involutive ideal of $S(B)$.

In the statement of the next corollary we retain the notation of Theorem 2.4. The proof is a mere combination of what has gone before.

2.5. COROLLARY. *Let B be a regular ring. Then:*

- (1) $\phi^{-1}(\bar{I})$ is a graded involutive ideal of $S(B)$.
- (2) \bar{I} is extended from $\phi^{-1}(\bar{I})$, that is, \bar{I} is generated by $\phi(\phi^{-1}(\bar{I}))$.

The main ingredient of Theorem 2.4 is the existence of a well-behaved Poisson bracket, which follows from the isomorphism between $S(A)$ and $\text{gr } \mathcal{A}(A)$. A similar isomorphism holds when A is the algebra of formal power series. For a set of hypotheses that cover both affine algebras and power series see [8, Sects. 1.1.2 and 1.1.3].

3. THE GLOBAL CASE

Let X be an irreducible smooth complex algebraic variety of dimension n . The sheaf of rings of differential operators \mathcal{D}_X is defined on an open affine subset U of X by

$$\mathcal{D}_X(U) = \mathcal{A}(\mathcal{A}(U)),$$

where $\mathcal{A}(U) = \Gamma(U, \mathcal{O}_X)$ is the coordinate ring of U . Since $\mathcal{D}_X(U)$ is a filtered K -algebra, it follows that \mathcal{D}_X is a filtered sheaf of K -algebras. Its associated graded sheaf will be denoted by $\text{gr } \mathcal{D}_X$. This sheaf has a very neat geometrical interpretation. Let T^*X be the cotangent bundle of X , and let $\pi: T^*X \rightarrow X$ be the canonical projection. Then

$$\text{gr } \mathcal{D}_X \cong \pi_* (\mathcal{O}_{T^*X}).$$

Following [1, Chap. VI, Sect. 1] we shall say that an open affine subset U of X has a *system of coordinates* if there exist $x_1, \dots, x_n \in \mathcal{A}(U)$ such that the differentials dx_1, \dots, dx_n form a basis of the module $\Omega^1(U)$ of Kähler differentials. The corresponding derivatives $\partial_1, \dots, \partial_n \in \text{Der}_{\mathbb{C}}(\mathcal{A}(U))$ satisfy $\partial_i(x_j) = \delta_{ij}$. Since X is smooth, each one of its points has a neighbourhood which admits a system of coordinates. If U has a system of coordinates, then $T^*X|_U$ is trivial and, in the notation of Sect. 1,

$$\mathcal{A}(T^*X|_U) \cong S(\mathcal{A}(U)) \cong \mathcal{A}(U)[\xi_1, \dots, \xi_n]$$

is a polynomial ring. Let α be the 1-form on $T^*X|_U$ defined by

$$\alpha = \sum_{i=1}^n \xi_i dx_i.$$

If $f_1, f_2 \in \mathcal{A}(T^*X)$, then a calculation shows that

$$n df_1 \wedge df_2 \wedge (d\alpha)^{n-1} = \{f_1, f_2\} (d\alpha)^n,$$

where $\{f_1, f_2\}$ denotes the Poisson bracket defined in Sect. 1. However, the form α is invariant under change of systems of coordinates of U . Let U_1, \dots, U_k be a cover of X by open sets, each of which admits a system of coordinates. Then there exists a 1-form ω , defined on the whole of T^*X and such that $\omega|_{U_j} = \alpha$, for $j = 1, \dots, k$.

We say that a subvariety W of T^*X is *involutive* (or *coisotropic*) if the ideal of W over any affine set U with a system of coordinates is involutive for the Poisson bracket defined by $\omega|_U$. Thus involutivity is a local property. It is not difficult to show that an involutive variety has dimension $\geq n$. For a more general approach to lagrangian varieties see [5].

We will see many natural examples of involutive varieties in the next section. Let us first prove a global version of Corollary 2.4. Let Y be a smooth subvariety of X . Then $T^*X|_Y$ is a subvariety of T^*X . Denote by

$$\rho: T^*X|_Y \rightarrow T^*Y$$

the natural projection. When X is affine, this map is the Spec of the homomorphism ϕ of Sect. 2. Recall that a subvariety of T^*X is *conical* if it is invariant under homotheties on the fibers of the cotangent bundle.

3.1. THEOREM. *Let W be a conical, involutive, irreducible subvariety of T^*X and assume that $\pi(W) \subseteq Y$, where Y is a smooth hypersurface of X . Then:*

- (1) $\rho(W)$ is a conical, involutive, irreducible subvariety of T^*Y .
- (2) $\rho^{-1}(\rho(W)) = W$ and $\dim \rho(W) = \dim W - 1$.

Proof. First note that since $\pi(W) \subseteq Y$, then $W \subseteq T^*X|_Y$. Moreover, because W is conical we also have that $\pi(W) = W \cap T_X^*X$ is a closed subvariety of X , where T_X^*X denotes the zero section of the bundle T^*X .

To prove the theorem it is enough to find an open cover of Y by affine sets $\{U_i\}$ of X such that (1) and (2) hold for $W \cap T^*U_i$ at each open set U_i of this cover. Since $Y \subset X$ is a smooth hypersurface, given $y \in Y$, there exists an affine open set $U \subseteq X$, $y \in U$, and x_1, \dots, x_n local parameters at y on X such that Y is given by $x_n = 0$ on U [10, Chap. II, Sect. 3.2, Theorem 5]. By choosing a smaller U , if necessary, we can assume that dx_1, \dots, dx_n are linearly independent at every $u \in U$. Therefore, $(x_1, \dots, x_n; \partial_1, \dots, \partial_n)$ form a system of "local coordinates" at y in U [1, Chap. VI, Sect. 1]. Thus it is enough to prove the theorem under the additional hypothesis that X is affine and has a system of coordinates x_1, \dots, x_n such that the equation of Y in X is $x_n = 0$.

Since this is a system of coordinates, there exists a derivation ∂_n of $\mathcal{A}(X)$ such that $\partial_n(x_n) = 1$. But $\mathcal{A}(T^*Y) \cong S(\mathcal{A}(Y))$ in the notation of Sect. 1. Thus ρ gives rise to a map

$$\rho^*: S(\mathcal{A}(Y)) \rightarrow \frac{S(\mathcal{A}(X))}{S(\mathcal{A}(X))_{x_n}} = \overline{S(\mathcal{A}(X))}$$

and $\rho^* = \phi$. If I is the ideal of W in $\mathcal{A}(T^*X)$, then, by Corollary 2.5,

$$\bar{I} = (\bar{I} \cap S(\mathcal{A}(Y)))\overline{S(\mathcal{A}(X))}$$

and $\bar{I} \cap S(\mathcal{A}(Y))$ is an involutive graded prime ideal of $S(\mathcal{A}(Y))$. Therefore, $\rho(W)$ is a conical, involutive, irreducible subvariety of T^*Y and

$$W = W \cap T^*X|_Y = \rho^{-1}(\rho(W)),$$

as required. ■

The most important special case of Theorem 3.1 is the following

3.2. COROLLARY. *Let W be a conical, irreducible, involutive subvariety of T^*X . If $Y = \pi(W)$ is a smooth subvariety of X , then*

- (1) $\rho(W)$ is a conical, irreducible, involutive, subvariety of T^*Y of dimension $\dim W - \text{codim}(Y, X)$.
- (2) $\rho^{-1}(\rho(W)) = W$.

Proof. Once again it is enough to prove the result locally. Then, we can suppose that X is affine and Y is given by $x_1 = 0, \dots, x_k = 0$ in X , where $k = \text{codim}(Y, X)$ [10, Chap. II, Sect. 3.2, Theorem 5].

Let $Y' \subset X$ be given by $x_1 = 0, \dots, x_{k-1} = 0$. Then Y' is nonsingular [10, Chap. II, Sect. 3.2, Theorem 4] and $Y \subset Y' \subset X$. Moreover $\text{codim}(Y, Y') = 1$ and $\text{codim}(Y', X) = k - 1$. Let

$$\varphi: T^*X|_{Y'} \rightarrow T^*Y'$$

and

$$\psi: T^*Y'|_Y \rightarrow T^*Y$$

be the canonical maps. Since $T^*X|_Y \subset T^*X|_{Y'}$, we denote by

$$\varphi|_Y: T^*X|_Y \rightarrow T^*Y'|_Y$$

the restriction of φ to $T^*X|_Y$. Note that

$$\rho = \psi \circ \varphi|_Y.$$

The result follows by Theorem 3.1 and induction on k . \blacksquare

4. APPLICATIONS

Let X be an irreducible smooth complex variety of dimension n . Let \mathcal{M} be a coherent left \mathcal{D}_X -module. If U is an affine open set of X , then $\mathcal{M}(U) = \Gamma(U, \mathcal{M})$ has a good filtration \mathcal{F} ; see [1, Chap. VI, Sect. 1.9]. In other words $\text{gr}^{\mathcal{F}} \mathcal{M}(U)$ is a finitely generated module over $\mathcal{A}(T^*X|_U) \cong \text{gr} \mathcal{A}(U)$. Let $I(\mathcal{M}(U))$ be the radical of the annihilator of $\text{gr}^{\mathcal{F}} \mathcal{M}(U)$ in $\mathcal{A}(T^*X|_U)$. This ideal is independent of the good filtration used to calculate it. Thus it is an invariant of $\mathcal{M}(U)$, called its *characteristic ideal*. The set of zeros of $I(\mathcal{M}(U))$ in $T^*X|_U$ is called the *characteristic variety* of $\mathcal{M}(U)$. It will be denoted by $\text{Ch}(\mathcal{M}(U))$.

The characteristic variety $\text{Ch}(\mathcal{M}(U))$ is a conical subvariety of $T^*X|_U$, which is also involutive, in the sense of Sect. 3. An algebraic proof of this result was given by O. Gabber in [2].

We may glue the characteristic varieties of sections of \mathcal{M} over an open affine cover. The variety so obtained is the *characteristic variety* of \mathcal{M} . Since involutivity is a local property, we conclude that the characteristic variety $\text{Ch}(\mathcal{M})$ is a conical involutive subvariety of T^*X . The support of the sheaf \mathcal{M} can be calculated from the characteristic variety:

$$\text{Supp}(\mathcal{M}) = \text{Ch}(\mathcal{M}) \cap T^*_X X = \pi(\text{Ch}(\mathcal{M})),$$

where $T^*_X X$ is the zero section of T^*X .

Our first application makes use of a famous theorem of Kashiwara, see [1, Chap. VI, Sect. 7] and [4]. Let $\mu(X)$ be the category of coherent left \mathcal{D}_X -modules. If Y is a subvariety of X , we will denote by $\mu_Y(X)$ the full subcategory of coherent left \mathcal{D}_X -modules with support contained in Y .

4.1. THEOREM (Kashiwara's equivalence). *Let Y be a smooth subvariety of X and let $i: Y \hookrightarrow X$ be the natural embedding. The \mathcal{D} -module theoretic direct image functor*

$$i_+ : \mu(Y) \rightarrow \mu_Y(X)$$

is an equivalence of categories. If \mathcal{M} is an object of $\mu(Y)$ then $\text{Ch}(i_+(\mathcal{M})) = \rho^{-1}(\text{Ch}(\mathcal{M}))$.

Corollary 3.2 can be considered as a geometrical version of Kashiwara's theorem. Theorem 4.1 deals with \mathcal{D}_X -modules and shows how the corresponding characteristic varieties are related. Corollary 3.2 extends the conclusion to all involutive conical varieties. Our first application is obtained by combining the two.

4.2. THEOREM. *Let W be a conical, irreducible, involutive subvariety of T^*X and suppose that $Y = \pi(W)$ is smooth in X . If $\rho(W)$ is the characteristic variety of a coherent \mathcal{D}_Y -module, then W is the characteristic variety of a coherent \mathcal{D}_X -module.*

Proof. Suppose that $\rho(W) = \text{Ch}(\mathcal{M})$, where \mathcal{M} is a coherent \mathcal{D}_Y -module. By Theorem 4.1

$$\text{Ch}(i_+(\mathcal{M})) = \rho^{-1}(\rho(W)).$$

Since $\pi(W) = Y$, we conclude from Corollary 3.2 that

$$\text{Ch}(i_+(\mathcal{M})) = W. \quad \blacksquare$$

We are now ready to return to the problem stated in the Introduction, namely: Which conical involutive subvarieties of T^*X are characteristic varieties of a coherent \mathcal{D}_X -module? We consider some special cases. For the next two examples, W is an involutive irreducible conical subvariety of T^*X and $\pi(W) = Y$ is a smooth subvariety of X .

4.3. EXAMPLE. Suppose that X is an affine variety and that $\dim X + \dim Y - \dim W = 1$. In this case,

$$\text{codim}(\rho(W), T^*Y) = \dim X + \dim Y - \dim W = 1.$$

Hence $\rho(W)$ is a hypersurface in T^*Y . Since Y is affine, there exists $f \in \mathcal{A}(T^*Y)$ such that $\rho(W) = \mathcal{Z}(f)$. Choose $d \in \mathcal{D}(Y)$ such that f is the symbol of d in $\text{gr } \mathcal{D}(Y) \cong \mathcal{A}(T^*Y)$. Put $M = \mathcal{D}(Y)/\mathcal{D}(Y)d$. Then $\text{Ch}(M) = \rho(W)$. By Theorem 4.2, W is a characteristic variety.

The dimension of an involutive subvariety of T^*X cannot be less than n . Involutive varieties of dimension n are very important in symplectic geometry; they are called *lagrangian* subvarieties. Equivalently, a subvari-

ety W of T^*X is lagrangian if the restriction of $d\omega$ to W is zero. A \mathcal{D}_X -module whose characteristic variety is lagrangian is called *holonomic*. These modules play a vital role in \mathcal{D} -module theory.

4.4. EXAMPLE. Suppose that W is lagrangian. By Corollary 3.2, $\rho(W)$ is involutive and has dimension $n - \text{codim}(Y, X) = \dim Y$. Hence $\rho(W)$ is an irreducible lagrangian subvariety of T^*Y . Since

$$\pi(\rho(W)) = \pi(W) = Y$$

and $\rho(W)$ is irreducible, we conclude that $\rho(W) = T_Y^*Y$, the zero section of T^*Y . Hence $\rho(W) = \text{Ch}(\mathcal{O}_Y)$. It follows by Theorem 4.2 that W is a characteristic variety.

If $\pi(W)$ is not smooth, then the conclusion of Example 4.4 must be weakened.

4.5. THEOREM. *Let W be a conical, irreducible, involutive subvariety of T^*X . If W is lagrangian, then W is an irreducible component of the characteristic variety of a holonomic \mathcal{D}_X -module*

Proof. Let U be an open set of X such that $V = U \cap Y$ is the set of regular points of Y . Let

$$\rho: T^*U|_V \rightarrow T^*V$$

be the canonical map. Since $W \cap T^*U$ is involutive, so is $\rho(W \cap T^*U)$ by Corollary 3.2. Assume that W is lagrangian.

By Corollary 3.2 again, $\rho(W \cap T^*U)$ has dimension equal to $\dim V$. Hence $\rho(W \cap T^*U)$ is lagrangian. Since

$$\pi(\rho(W \cap T^*U)) = V$$

we conclude that $\rho(W \cap T^*U) = T_V^*V$. Hence $\rho(W \cap T^*U) = \text{Ch}(\mathcal{O}_V)$ and, as in Theorem 4.2,

$$\text{Ch}(i_+(\mathcal{O}_V)) = W \cap T^*U$$

where $i: V \hookrightarrow U$ is the closed embedding. Denoting by $j: U \hookrightarrow X$ the open embedding, we have that

$$W \cap T^*U \subseteq \text{Ch}(j_+(i_+(\mathcal{O}_V))).$$

Since W is the closure of $W \cap T^*U$, we conclude that $W \subseteq \text{Ch}(j_+(i_+(\mathcal{O}_V)))$. But holonomy is preserved by direct images. Hence $\dim \text{Ch}(j_+(i_+(\mathcal{O}_V))) = n$. Thus W is an irreducible component of $\text{Ch}(j_+(i_+(\mathcal{O}_V)))$. ■

Unfortunately this is the best one can do in the general case, as shown by the example in the next section.

5. AN EXAMPLE

In this section we give an example of an irreducible lagrangian variety which is not a characteristic variety. The example is well known to the experts, but does not seem to have appeared in print before. We include it here for the sake of completeness.

Let $X = \mathbb{A}^n(\mathbb{C})$ be the affine n -space with coordinate functions x_1, \dots, x_n . Put $f = x_1^2 + \dots + x_n^2$ and let V be the variety $f = 0$. This is the *quadratic cone* in X , and it is nonsingular except at the origin. In other words, if $X' = X \setminus \{0\}$, then $V' = V \cap X'$ is nonsingular subvariety of X' . Now let W be the closure in T^*X of the conormal bundle of V' in T^*X' . This is sometimes called the *conormal variety* of V in T^*X . We will denote by π the canonical projection of T^*X on X .

Let ξ_1, \dots, ξ_n be the coordinate functions of the fibers of T^*X . The coordinate ring of T^*X will be identified with the polynomial ring in the variables x_1, \dots, x_n and ξ_1, \dots, ξ_n . Let J be the ideal of this polynomial ring generated by f , $\sum_1^n x_i \xi_i$, $\sum_1^n \xi_i^2$, and by

$$x_i \xi_j - x_j \xi_i,$$

for $1 \leq i < j \leq n$.

5.1. LEMMA. *The variety of zeros of J is equal to W .*

Proof. Let U be the open set $T^*X \setminus T_0^*X$, where T_0^*X is the fiber at 0 of the cotangent bundle. A point of $W \cap U$ is determined by the equations

$$\xi_1 = \lambda x_1, \dots, \quad \xi_n = \lambda x_n,$$

where λ is a nonzero complex number. It easily follows from these equations that

$$W \cap U = \overline{\mathcal{A}(J)} \cap U.$$

Since W is, by definition, the closure of $W \cap U$ in T^*X , it follows that it is an irreducible component of $\overline{\mathcal{A}(J)}$. Let Y be another irreducible component of $\overline{\mathcal{A}(J)}$. Then we must have that $Y \subseteq T_0^*X$, the fiber of 0 in T^*X . But J is an involutive ideal. Hence Y must be an involutive variety, and consequently $\dim Y = n$. However, $\sum_1^n \xi_i^2 \in J$, and so $\dim Y < n$, a contradiction. Thus $\overline{\mathcal{A}(J)} = W$, as required. ■

We now turn to the modules. Let i stand for the inclusion $V \hookrightarrow X$ and its restrictions to open sets of X . Denote by X_i the open set with equation $x_i \neq 0$ in X . Put $V_i = V \cap X_i$. Since V_i is nonsingular, we can apply Kashiwara's equivalence to conclude that

$$i_+(\mathcal{Q}(V_i))$$

is an irreducible $\mathcal{A}(X_i)$ -module. It is easy to give an explicit description of this module. The generators of the module of derivations of $\mathcal{A}(V_i)$ are

$$D_j = \partial_j - \frac{x_j}{x_i} \partial_i$$

for $i \neq j$. Hence we have an isomorphism of $\mathcal{A}(V_i)$ -modules

$$\mathcal{A}(V_i) \cong \mathcal{A}(V_i) \left/ \sum_{j \neq i} \mathcal{A}(V_i) D_j \right.$$

On the other hand $\mathcal{A}(X_i)$ is generated by $\mathcal{A}(X_i)$ and the derivations D_j , for $j \neq i$, together with

$$d = \frac{1}{x_i} \partial_i.$$

An easy calculation shows that

$$i_+(\mathcal{A}(V_i)) \cong \mathbb{C}[d] \otimes_{\mathbb{C}} \mathcal{A}(V_i) \left/ \sum_{j \neq i} \mathcal{A}(V_i) D_j \right.,$$

where d and f act only on the first factor of the tensor product and $f \cdot d = -2$ in $\mathbb{C}[d]$. For more details see [1, p. 259].

In [3] it is proved that the module of derivations of V is generated by

$$\Delta_{ij} = x_i \partial_j - x_j \partial_i,$$

for $1 \leq i < j \leq n$, and by the Euler operator

$$E = \sum_1^n x_i \partial_i.$$

The Euler operator can be written in $\mathcal{A}(X_i)$ in the form

$$E = \sum_{j \neq i} x_j D_j + fd.$$

Applying E to the element $1 \otimes 1$ of $i_+(\mathcal{A}(V_i))$, we get that

$$E(1 \otimes 1) = fd \otimes 1 + \sum_{j \neq i} 1 \otimes x_j D_j = -2(1 \otimes 1).$$

Hence $E + 2$ annihilates $1 \otimes 1$. Since $\Delta_{ij} = x_i D_j$ for $i < j$, it follows that Δ_{ij} annihilates $1 \otimes 1$. This suggests that we should consider the $\mathcal{A}(X)$ -module

$$M = \mathcal{A}(X)/L,$$

where L is the left ideal of $\mathcal{A}(X)$ generated by $E + 2$, f , and Δ_{ij} , for $1 \leq i < j \leq n$.

5.2. THEOREM. *If $n \geq 5$ is an odd integer, then*

- (1) M is an irreducible A_n -module.
- (2) $\text{Ch}(M) = W \cup T_0^*X$.
- (3) $\text{Supp}(M) = V$.

Proof. A straightforward calculation shows that

$$\mathcal{A}(X_i) \otimes_{\mathcal{A}(X)} \mathcal{A}(X)/L \cong i_+(\mathcal{A}(V_i))$$

as $\mathcal{A}(X_i)$ -modules. In particular, $\mathcal{A}(X_i) \otimes M$ is an irreducible $\mathcal{A}(X_i)$ -module. Suppose that L' is a left ideal of $\mathcal{A}(X)$ that contains L properly. Then

$$\mathcal{A}(X_i) \otimes_{\mathcal{A}(X)} \mathcal{A}(X)/L' = 0,$$

for $1 \leq i \leq n$. Therefore $\text{Supp}(\mathcal{A}(X)/L') = \{0\}$. Thus, there exists a positive integer k such that $x_i^k \in L'$ for $1 \leq i \leq n$. But

$$kx_jx_i^{k-1} = [\Delta_{ij}, x_i^k] \in L',$$

for $i \neq j$, and so

$$(k + n - 3)x_i^{k-1} = \partial_i \cdot x_i^k + \sum_{j \neq i} \partial_j \cdot x_jx_i^{k-1} - x_i^{k-1}(E + 2)$$

belongs to L' . Since $n \geq 5$, we conclude that $x_i^{k-1} \in L'$. By induction on k we have that $1 \in L'$ and so $L' = \mathcal{A}(X)$. Thus L is a maximal left ideal of $\mathcal{A}(X)$ and M is an irreducible $\mathcal{A}(X)$ -module.

Let us calculate the characteristic variety of M . We have seen that if U_i is the open set $x_i \neq 0$ in T^*X , then

$$\text{Ch}(M) \cap U_i = \text{Ch}(\mathcal{A}(X_i) \otimes_{\mathcal{A}(X)} M) = W \cap U_i.$$

Hence W is an irreducible component of $\text{Ch}(M)$. It also follows from the above equation that if $\text{Ch}(M)$ has another irreducible component then it must be equal to T_0^*X .

If $\text{Ch}(M) = W$, then $\sum_1^n \xi_i^2 \in \text{rad}(\sigma(L))$. We will show that this cannot happen. Let $\theta = \sum_1^n \partial_i^2$. It is easy to prove, by induction on k , that

$$[\theta^k, f] = 4k\theta^{k-1}E + 2k(n - 2(k - 1))\theta^{k-1}.$$

Therefore

$$[\theta^k, f] \equiv 2k(n - 2k - 2)\theta^{k-1} \pmod{L}. \tag{5.3}$$

If $n \geq 5$ is odd, then $n - 2(k + 1) \neq 0$ for all k . Hence if $\sigma(L)$ contains an element whose leading term is $\sigma(\theta)^k$, then it contains $\sigma(\theta)$. Thus we can assume that L contains an element of the form

$$\theta + \sum_1^n g_i \partial_i + h,$$

where $g_1, \dots, g_n, h \in \mathbb{C}[x_1, \dots, x_n]$. Commuting this element with f and using (5.3), we get that

$$2(n - 4) + 2 \sum_1^n g_i x_i \in L.$$

Hence $(n - 4) + \sum_1^n g_i x_i \in \sigma(L)$. But we are assuming that $\text{Ch}(M) = W$. Thus $\pi(W) = V$, and we have that

$$\text{rad}(\sigma(L)) \cap \mathbb{C}[x_1, \dots, x_n] = (f),$$

the ideal of $\mathbb{C}[x_1, \dots, x_n]$ generated by f . Since f is homogeneous of degree 2 and $n \geq 5$, the ideal (f) cannot contain $(n - 4) + \sum_1^n g_i x_i$, a contradiction. Hence $\text{Ch}(M) \neq W$, and so $\text{Ch}(M) = W \cup T_0^*X$. In particular $\text{Supp}(M) = V$. ■

We are now ready to prove that W is only an irreducible component of the characteristic variety of the module constructed in Theorem 4.5. Let $j: X' \hookrightarrow X$ be the standard open embedding.

5.4. PROPOSITION. *The characteristic variety of $j_+i_+(Q_{V'})$ contains $W \cup T_0^*X$.*

Proof. Let U be an open set of X . Then

$$\begin{aligned} \Gamma(U, j_+i_+(Q_{V'})) &\cong \Gamma(U \cap X', i_+(Q_{V'})) \\ &\cong [\mathcal{L}(U \cap X') / \mathcal{L}(U \cap X')f] \otimes \mathcal{O}(V' \cap U) \end{aligned}$$

considered as a $\mathcal{L}(U)$ -module. Thus the element corresponding to $1 \otimes 1$ is a nonzero global section of $j_+i_+(Q_{V'})$. This section is annihilated by f , $E + 2$, and Δ_{ij} for $1 \leq i < j \leq n$. Hence there is a nonzero map

$$M \rightarrow \Gamma(X, j_+i_+(Q_{V'})).$$

Since M is irreducible by Theorem 5.2, it is a submodule of $\Gamma(X, j_+i_+(Q_{V'}))$. Therefore

$$\text{Ch}(M) \subseteq \text{Ch}(j_+i_+(Q_{V'})),$$

since X is affine. ■

It turns out that it is not merely a question of the previous construction being inefficient; the variety W is not characteristic variety of any $\mathcal{D}(X)$ -module. We will prove this in the next theorem. The proof depends on switching from algebraic to analytic \mathcal{D} -modules. Let X^a denote the analytic space corresponding to the variety X . If M is a coherent left \mathcal{D}_X -module, then

$$\text{Ch}(\mathcal{D}_{X^a} \otimes_{i^{-1}\mathcal{D}_X} i^{-1}M) = \text{Ch}(M)^a,$$

where $i: X^a \rightarrow X$ is the natural continuous map. Thus if W is a characteristic variety of some $\mathcal{D}(X)$ -module, then W^a is the characteristic variety of some \mathcal{D}_{X^a} -module. This is our starting point. Note that Theorem 5.2(1) and Proposition 5.4 hold for analytic \mathcal{D} -modules with exactly the same proof.

5.5. THEOREM. *The lagrangian variety W is not the characteristic variety of a $\mathcal{D}(X)$ -module.*

Proof. As we have noted above it is enough to prove that W^a is not a characteristic variety. Throughout the proof we will work in the analytic category, but we will omit the superscript a from the notation to make it easier to digest.

Let \mathcal{N} be a coherent left \mathcal{D}_X -module whose characteristic variety is W . Since W is irreducible and \mathcal{N} is holonomic, we may assume that \mathcal{N} is irreducible. Let

$$\mathcal{N} = \mathcal{D}_{X'} \otimes \mathcal{N}.$$

Note that $\mathcal{N} \neq 0$, because \mathcal{N} is not supported at zero. Moreover, since \mathcal{N} is irreducible, so is \mathcal{N}' ; see, for example, [1, 10.8(a)]. But $\text{Supp}(\mathcal{N}') = V'$. Hence, by the equivalence of Theorem 4.1, we have that \mathcal{N}' is extended from an irreducible $\mathcal{D}_{V'}$ -module. However, V' is simply connected. In fact, V' is homotopy equivalent to $K = V \cap S_\epsilon$, where S_ϵ is the sphere consisting of all $z \in \mathbb{C}^n$ with $\|z\| = \epsilon$, $\epsilon > 0$ (the maps $f: V' \rightarrow K$, $f(z_1, \dots, z_n) = \epsilon \|z\|^{-1}(z_1, \dots, z_n)$, and the inclusion $i: K \hookrightarrow V'$ are easily seen to establish a homotopy equivalence). But K is simply connected, if $n \geq 4$ by [9, Theorem 5.2]. Thus by [1, Chap. IV, Sect. 1] we conclude that

$$\mathcal{N}' \cong i_+(\mathcal{Q}_{V'}).$$

Now the canonical map

$$\mathcal{N} \rightarrow j_+i_+(\mathcal{Q}_{V'}),$$

gives rise to

$$\Gamma(X, \mathcal{N}) \rightarrow \Gamma(X, j_+i_+(\mathcal{Q}_{V'})).$$

This last map is nonzero because $X = \mathbb{C}^n$ is a Stein manifold.

Since $\Gamma(X, \mathcal{M})$ is irreducible, we conclude that $\Gamma(X, j_+ i_+(Q_{V'}))$ contains an irreducible submodule isomorphic to $\Gamma(X, \mathcal{M})$; we will call it F . On the other hand, arguing as in Proposition 5.4, we conclude that $\mathcal{L}(X)/\mathcal{L}(X)L$ is an irreducible submodule of $\Gamma(X, j_+ i_+(Q_{V'}))$, which will be denoted by G .

However,

$$\Gamma(X_i, j_+ i_+(Q_{V'})) = \Gamma(X_i, i_+(Q_{V'}))$$

is an irreducible $\mathcal{L}(X_i)$ -module. Since $\text{Supp}(F) = \text{Supp}(G) = V$, we also have that

$$\mathcal{L}(X_i) \otimes F = \mathcal{L}(X_i) \otimes G = \Gamma(X_i, j_+ i_+(Q_{V'})).$$

Thus $\text{Supp}((F + G)/G) = \{0\}$, and so $F \cap G \neq \{0\}$. Since both F and G are irreducible, we conclude that $F = G$. Thus

$$W \cup T_0^* X = \text{Ch}(\mathcal{L}(X)/\mathcal{L}(X)L) = \text{Ch}(G) = \text{Ch}(F) \subseteq \text{Ch}(\mathcal{M}),$$

which is a contradiction. ■

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