A DICRITICAL FOLIATION WITH ONE SINGULARITY AND NO ALGEBRAIC SOLUTIONS

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Abstract. We give an example of a family of dicritical singular holomorphic foliations of the projective plane, with one singular point, whose very generic elements do not have any algebraic solutions.

1. Introduction

In 1979 J.-P. Jouanolou published his now famous example of a singular holomorphic foliation of the complex projective plane $\mathbb{P}^2$ without algebraic solutions [12]. Since then many other examples of such foliations have been found, see, for example, [16] and [6]. The foliations in these examples are, most of them, non-degenerate and non-dicritical.

More recently there has been a flurry of papers on holomorphic foliations of $\mathbb{P}^2$ with one singularity. Some of these papers have presented examples of holomorphic foliations with one singularity and no algebraic solutions, as is the case of [1], [2], [10], and [7]. However, in all these examples the foliations are non-dicritical.

The main result of this paper is Theorem 4.6, which states that the very generic element of a family of dicritical holomorphic foliations with one singularity, to be defined in Section 3, does not have any algebraic solutions. As part of the proof of this theorem, we show, in Section 4, that the family contains a foliation without any algebraic solutions. This requires that we find an upper bound for the degree of the algebraic solutions of the foliation. The strategy we use to do this can be applied to other examples, as shown in [8].

The paper is structured as follows. In Section 2 we collect a number of basic results on singular holomorphic foliations of $\mathbb{P}^2$. The family of dicritical singular holomorphic foliations is defined in Section 3, where we also prove part of Theorem 4.6, except for the analysis of the example of a foliation without algebraic solutions, which is the content of Section 4. Finally, it should be pointed out that several results in this paper require computer assisted proofs, which were obtained with the help of the computer algebra system Axiom [9].

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2. Preliminaries

In this Section we collect a number of basic results on singular holomorphic foliations that will be used throughout the paper. Given a polynomial $f \in \mathbb{C}[x, y]$ and a non-negative integer k, we denote by f_k the homogeneous component of degree k of f. The *order* $\nu_p(f)$ of f at $p = (x_0, y_0) \in \mathbb{C}^2$ is the smallest $k \geq 0$ for which the homogeneous component of degree k of $f(x+x_0, y+y_0)$ is non-zero. Both definitions will also be applied to power series in two variables when $p = (0, 0)$.

2.1. **Basic concepts.** A *singular holomorphic foliation* $\mathcal{F}$ over the complex projective plane $\mathbb{P}^2$ is defined by a 1-form $\Omega = Adx+Bdy+Cdz$, where $A, B, C \in \mathbb{C}[x, y, z]$ are homogeneous polynomials of degree $d + 1 \geq 1$ such that $xA + yB + zC = 0$. The 1-forms for which this condition holds are called *projective*.

A point $p \in \mathbb{P}^2$ is *singular* for $\mathcal{F}$ if $A(p) = B(p) = C(p) = 0$. By [12, Proposition 4.1, p. 126], all foliations of $\mathbb{P}^2$ have a singular point. The set of singular points of $\mathcal{F}$ will be denoted by $\mathrm{Sing}(\mathcal{F})$. By Bézout's Theorem, this set is finite if and only if $\gcd(A, B, C) = 1$. In this case the integer

$$d = \deg(A) - 1 = \deg(B) - 1 = \deg(C) - 1$$

is the *degree* of $\mathcal{F}$, which we will denote by $\deg(\mathcal{F})$.

From now on we will use the word foliation as an abbreviation of singular holomorphic foliation with a finite number of singularities. In particular, we always assume that, when the foliation is defined by a projective homogeneous 1-form $\Omega = Adx + Bdy + Cdz$, then $\gcd(A, B, C) = 1$.

If U is the open set of $\mathbb{P}^2$ whose points have non-zero z-coordinate and $U \hookrightarrow \mathbb{P}^2$ is the inclusion map, then $\mathcal{F}|_U$ is defined by the 1-form

$$\Omega|_U = A(x, y, 1)dx + B(x, y, 1)dy.$$

Conversely, a foliation can be defined by a 1-form $\omega = adx+bdy$, where $a, b \in \mathbb{C}[x, y]$ and $\gcd(a, b) = 1$. In order to do this, we take the pullback $\pi^*(\omega)$ of ω under the map $\pi : U \to \mathbb{C}^2$ defined by $\pi([x : y : z]) = (x/z, y/z)$ and multiply the resulting form by the smallest power of z that eliminates the poles. One easily checks that the resulting 1-form is projective. This form will be called the *projectivization of* ω. From now on we will identify $\mathbb{C}^2$ with the open set $z \neq 0$ of $\mathbb{P}^2$. If

$$e = \max\{\deg(a), \deg(b)\}$$

and $\mathcal{F}$ is the foliation of $\mathbb{P}^2$ defined by ω, then

$$\deg(\mathcal{F}) = \begin{cases} e - 1 & \text{when } xa_e + yb_e = 0, \\ e & \text{otherwise,} \end{cases}$$

where a_e and b_e denote the homogeneous components of degree e of a and b.

Let $U \cong \mathbb{C}^2$ be the open set $z \neq 0$ of $\mathbb{P}^2$ and let $\Omega|_U = adx + bdy$. The *algebraic multiplicity* of Ω at $p_0 = (0, 0)$ is the non-negative number $\nu(\Omega) = \min\{\nu_{p_0}(a), \nu_{p_0}(b)\}$. When $xa_\nu + yb_\nu = 0$, the singularity p_0 is said to be *dicritical*. A foliation is *dicritical* if it has at least one dicritical singularity.

Let $F \in \mathbb{C}[x, y, z]$ be a non-constant square-free homogeneous polynomial. The algebraic curve

$$\mathrm{V}(F) = \{p \in \mathbb{P}^2 \mid F(p) = 0\}$$

is an *algebraic solution* of the foliation $\mathcal{F}$ if $\Omega \wedge dF = F\eta$ for some polynomial 2-form η with homogeneous coeffcients and degree d. If $F \neq z$ and U is the open set $z \neq 0$, then $\omega = \Omega|_U = adx + bdy$, and the previous equation is equivalent to

$$\omega \wedge df = gf dx \wedge dy, \tag{2.1}$$

where $f = F(x, y, 1)$ and $g \in \mathbb{C}[x, y]$. The polynomial g is the *co-factor* of f, and its degree is less than or equal to d. In this case we often write simply that f is an algebraic solution of $\mathcal{F}$ with co-factor g. Setting $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$, equation (2.1) can also be written as $D(f) = gf$, where $D = -b\partial_x + a\partial_y$ is the *dual derivation* of ω. Moreover, by [13, Theorem 2.1, p. 13] if f is an algebraic solution of ω then so are its irreducible factors. In particular, if a foliation has an algebraic solution, then it has one that is irreducible.

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ defined by the homogeneous projective 1-form Ω. A projective transformation σ is an *isotropy* of $\mathcal{F}$ if $\sigma^*(\Omega) = \lambda\Omega$, for some non-zero complex number λ. When this happens we also say that $\mathcal{F}$ is *invariant* under σ. The set of all isotropies of $\mathcal{F}$, denoted by $\mathrm{iso}(\mathcal{F})$, is a group.

By abuse of notation we will denote by σ both, a projective transformation, and its restriction to the open set $z \neq 0$ of $\mathbb{P}^2$. Thus, given a polynomial $f \in \mathbb{C}[x, y]$ it makes sense to compute the pullback $\sigma^*(f)$, which we will also denote by f^σ. When $f^\sigma = \mu f$, for some $\mu \in \mathbb{C} \setminus \{0\}$, the polynomial f is said to be *semi-invariant* under σ.

Lemma 2.1. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ defined on $\mathbb{C}^2$ by the 1-form ω. If σ is an isotropy of finite order of a foliation $\mathcal{F}$ and $\phi \in \mathbb{C}[x, y]$ is an irreducible algebraic solution of $\mathcal{F}$, then there exists $k > 0$ such that $f = \phi \cdot \phi^\sigma \cdots \phi^{\sigma^k}$ is a square-free algebraic solution of $\mathcal{F}$ that is semi-invariant under σ. Moreover, if g is the co-factor of f and $\sigma^*(\omega) = \lambda\omega$, for some non-zero $\lambda \in \mathbb{C}^2$, then $\sigma^*(g) = \lambda \det(\sigma)^{-1} g$.*

Proof. Let ϕ be an algebraic solution of $\mathcal{F}$ with co-factor γ. Then,

$$\lambda\omega \wedge d(\phi^\sigma) = \sigma^*(\omega \wedge d\phi) = \sigma^*(\gamma\phi dx \wedge dy) = \gamma^\sigma \phi^\sigma \det(\sigma) dx \wedge dy. \tag{2.2}$$

By induction, ϕ^{σ^i} is an algebraic solution of $\mathcal{F}$ for all $i \geq 0$. Now let k be the smallest positive integer for which there exists a non-zero complex number μ such that $\phi^{\sigma^{k+1}} = \mu\phi$. For such k, the polynomial f in the statement of the lemma satisfies

$$\sigma^*(f) = \prod_{i=0}^{k} \sigma^*(\phi^{\sigma^i}) = \prod_{i=0}^{k} \phi^{\sigma^{i+1}} = \mu \prod_{i=0}^{k} \phi^{\sigma^i} = \mu f. \tag{2.3}$$

Thus, f is semi-invariant. It is also square-free because, if $\phi^{\sigma^m} = \nu\phi^{\sigma^\ell}$, for non-negative integers $m > \ell$ and $\nu \in \mathbb{C} \setminus \{0\}$, then

$$\phi^{\sigma^{m-\ell}} = \nu\phi;$$

so that, $m - \ell \geq k$, by our choice of k. Moreover, by [13, Lemma 1.6, p. 12], f is also an algebraic solution of $\mathcal{F}$. Finally, replacing ϕ with f and γ with g in (2.2), it follows from (2.3) that $g^\sigma = \lambda \det(\sigma)^{-1} g$. □

2.2. **Resolution of singularities.** In order to use a computer to show that a given foliation does not have any algebraic solutions, we need an upper bound on the degrees of these solutions. To this end we will use blow-ups to resolve the singularities of the foliation. The *blow-up* of $\mathbb{A}^2$ with centre at the origin is the surface

$$\mathcal{B} = \{((x,y),[u:v]) \in \mathbb{C}^2 \times \mathbb{P}^1 : xv = yu\}.$$

The *blow-up map* is the morphism $\phi : \mathcal{B} \to \mathbb{A}^2$ defined by $\phi((x,y),[u:v]) = (x,y)$. The surface $\mathcal{B}$ is the union of the open affine sets given by $u \neq 0$ and $v \neq 0$, both of which are isomorphic to $\mathbb{C}^2$. Assuming that $v \neq 0$ and setting $x_1 = u/v$ and $y_1 = y$, the isomorphism follows by rewriting the equation that defines $\mathcal{B}$ in the form $x = (u/v)y = x_1y_1$. Under this identification, the restriction of ϕ to $\mathcal{B}_{|v\neq 0}$, is the map

$$\phi_v : \mathbb{C}^2 \cong \mathcal{B}_{|v\neq 0} \longrightarrow \mathbb{C}^2$$

given by $\phi_v(x_1,y_1) = (x_1y_1,y_1)$. Similarly, denoting by x_2 and y_2 the coordinates of $\mathbb{C}^2 \cong \mathcal{B}_{|u\neq 0}$, the restriction of ϕ to this open set is the map ϕ_u defined by $\phi_u(x_2,y_2) = (x_2,x_2y_2)$.

Let $f \in \mathbb{C}[x,y]$ and consider the curve $C = \mathrm{V}(f) \subset \mathbb{C}^2$. If f has multiplicity $\nu = \nu_{p_0}(f)$ at $p_0 = (0,0)$, its *strict transform* $\widetilde{C}$ is the curve in $\mathcal{B}$ defined in the open sets $\mathcal{B}_{|v\neq 0}$ and $\mathcal{B}_{|u\neq 0}$ by the polynomials

$$f_v(x_1,y_1) = y_1^{-\nu}\phi_v^*(f) \;\text{ and }\; f_u(x_2,y_2) = x_2^{-\nu}\phi_u^*(f). \tag{2.4}$$

The strict transform of a foliation defined in $\mathbb{C}^2$ by the 1-form $\omega = adx + by$, with a singularity at the origin $p_0 = (0,0)$, can be similarly defined in $\mathcal{B}_{|v\neq 0}$ and $\mathcal{B}_{|u\neq 0}$ by the 1-forms $y_1^{-e}(\phi_v)^*(\omega)$ and $x_2^{-e}(\phi_u)^*(\omega)$, where

$$e = \begin{cases} \min\{\nu_{p_0}(a), \nu_{p_0}(b)\} & \text{when } p_0 \text{ is non-dicritical,} \\ \min\{\nu_{p_0}(a), \nu_{p_0}(b)\} + 1 & \text{when } p_0 \text{ is dicritical.} \end{cases}$$

Let ω be a 1-form with a singularity at the origin. This singularity is said to be *reduced* if at least one of the eigenvalues of the matrix

$$\begin{bmatrix} \partial b/\partial x & \partial b/\partial y \\ -\partial a/\partial x & -\partial a/\partial y \end{bmatrix}_{x=y=0} \tag{2.5}$$

is non-zero and, when both are non-zero, their ratio is not a positive rational number. The singularities of a holomorphic foliation can always be transformed, by a succession of blow-ups, into reduced singularities, see [14] or [11, p. 119-138].

We finish with a few facts about reduced singularities that will be used in later sections. Without loss of generality we may assume that the singularity is the origin. When both eigenvalues of (2.5) are non-zero, the singularity is said to be *non-degenerate*, when one of them is zero, it is called a *saddle-node*. By [4, p. 97], in both cases the foliation has two transversal separatrices at the singularity. If the singularity is a saddle-node, the separatrix tangent to the eigenvector with non-zero eigenvalue is the *strong* separatrix, while the one tangent to the eigenvector with null eigenvalue is the *weak separatrix*.

2.3. **Indices.** Let S be a germ of irreducible curve at a point $p_0 = (0,0) \in \mathbb{C}^2$, given locally by the vanishing of a power series $f = f_m + f_{m+1} + \cdots$, where m is the order of f. Its strict transform $\widetilde{S}$ is defined as in (2.4). By [5, Theorem 3.2.7, p. 73] since S is an irreducible germ, its strict transform meets E at only one point which we denote by $\widetilde{p_0}$. If $\widetilde{S}$ is non-singular and its intersection with E is transversal, then $\nu_{p_0}(S) = \nu_{\widetilde{p_0}}(\widetilde{S}) = 1$, by [5, Theorem 5.3.5, p. 83]. Thus, by Noether's Formula [5, Theorem 3.3.1, p. 79], if S_1 and S_2 are two germs of irreducible curves which intersect at p_0 and whose strict transforms are non-singular and transversal to E, then

$$(S_1 \cdot S_2)_{p_0} = \begin{cases} 1 & \text{if} \quad \widetilde{S}_1 \cap \widetilde{S}_2 = \emptyset, \\ 2 & \text{if} \quad \widetilde{S}_1 \cap \widetilde{S}_2 \neq \emptyset \text{ are transversal.} \end{cases}$$

Now, let $\mathcal{F}$ be a foliation defined in $\mathbb{C}^2$ by a 1-form $\omega = adx + bdy$ and let $f \in \mathbb{C}[x, y]$ be such that $C = \mathrm{V}(f) \subset \mathbb{C}^2$ is an algebraic solution of $\mathcal{F}$. Assuming that $f(0, 0) = 0$, it follows from [15, Lemma 2.4, p. 153] that there exist convergent power series g, h and a 1-form η, with power series coefficients, such that f and h are co-prime and

$$g\omega = hdf + f\eta, \tag{2.6}$$

in a neighbourhood of $(0, 0)$. Denoting the origin by p_0, the *GSV-index* $\mathcal{Z}(\mathcal{F}, C, p_0)$ is the vanishing order of $(h/g)|_C$ at p_0, see [3, p. 24]. If $C_1, \dots, C_k$ are the branches of C at p_0, then by [3, p. 38],

$$\mathcal{Z}(\mathcal{F}, C, p_0) = \sum_{i=1}^{k} \mathcal{Z}(\mathcal{F}, C_i, p_0) - 2 \sum_{1 \le i < j \le k} (C_i \cdot C_j)_{p_0}. \tag{2.7}$$

Note that k need not be equal to m, because C can have singular branches. In the next proposition we give the values of $\mathcal{Z}$ in some special cases. But, before we state it, we need a definition. The *Milnor number* of $\mathcal{F}$ at p_0 is

$$\mu(\mathcal{F}, p_0) = \dim_{\mathbb{C}} \left(\frac{\mathcal{O}_{p_0}}{(a, b)} \right),$$

where $\mathcal{O}_{p_0}$ is the localization of $\mathbb{C}[x, y]$ at the origin and (a, b) is the ideal of $\mathcal{O}_{p_0}$ generated by the coefficients a and b of ω. A proof of the next proposition can be found in [3, p. 39–40].

Proposition 2.2. *Let S be a germ of curve through $p_0 \in \mathbb{C}^2$ that is invariant under the germ of a one dimensional foliation $\mathcal{F}$. If $p_0 \notin \mathrm{Sing}(\mathcal{F})$ then $\mathcal{Z}(\mathcal{F}, S, p_0) = 0$. On the other hand, if $p_0 \in \mathrm{Sing}(\mathcal{F})$ is a saddle-node, then*

$$\mathcal{Z}(\mathcal{F}, S, p_0) = \begin{cases} 1 & \textit{if } S \textit{ is the strong separatrix at } p, \\ \mu(\mathcal{F}, p_0) + 1 & \textit{if } S \textit{ is the weak separatrix at } p. \end{cases}$$

3. The family

Given complex numbers

$$a_0,\ a_1,\ a_2,\ b_0 \ \text{ and } \ b_1,$$

let $\mathcal{G} = \mathcal{G}(a_0, a_1, a_2, b_0, b_1)$ be the foliation on $\mathbb{P}^2$ defined by the 1-form $adx + bdy$, where $a, b \in \mathbb{C}[x, y]$ are the polynomials whose monomials and respective coefficients are given in Tables 1 and 2.

Theorem 3.1. *If $a_2 \neq 0$ then $\mathrm{Sing}(\mathcal{G}) = \{[0 : 0 : 1]\}$ and this singularity is dicritical. Moreover, the line at infinity is not an algebraic solution of $\mathcal{G}$.*

Proof. Let $\Omega = Adx + Bdy + Cdz$ be the homogeneous projective 1-form that defines the foliation $\mathcal{G}$. Since the singularities of $\mathcal{G}$ are the common zeroes of A, B, and C, we apply Axiom's Gröbner Factorization Algorithm to the ideal I of

$$\mathbb{C}[a_0,\ a_1,\ a_2,\ b_0, b_1][x, y]$$

Monomials	Coefficients
x^4	4
xy	4
x^2y	$4b_0 + 12a_0$
x^3y	$4a_0b_0 + 12{a_0}^2$
x^4y	$b_1 + 4a_1 + 4{a_0}^3$
xy^2	$4b_1 + 12a_1$
x^2y^2	$(b_0 + 4a_0)b_1 + 8a_1b_0 + 24a_0a_1$
x^3y^2	$a_0b_0b_1 + 4a_0a_1b_0 + 2a_2 + 12{a_0}^2a_1$
y^3	$4a_2$
xy^3	${b_1}^2 + 8a_1b_1 + 6a_2b_0 + 16a_0a_2 + 12{a_1}^2$
x^2y^3	$a_0{b_1}^2 + (a_1b_0 + 4a_0a_1)b_1 + (2a_0a_2 + 4{a_1}^2)b_0 + 12{a_0}^2a_2 + 12a_0{a_1}^2$
y^4	$4a_2b_1 + 8a_1a_2$
xy^4	$a_1{b_1}^2 + \left(a_2b_0 + 4{a_1}^2\right)b_1 + 6a_1a_2b_0 + 16a_0a_1a_2 + 4{a_1}^3$
y^5	$a_2{b_1}^2 + 4a_1a_2b_1 + 2{a_2}^2b_0 + 8a_0{a_2}^2 + 4{a_1}^2a_2$

TABLE 1. Coefficients of a in $adx + bdy$.

Monomials	Coefficients
x^2	-4
x^3	$-12a_0$
x^4	$-12{a_0}^2$
x^5	$-b_1 - 4a_1 - 4{a_0}^3$
x^2y	$-12a_1$
x^3y	$-b_0b_1 - 4a_1\ b_0 - 24a_0a_1$
x^4y	$-a_0b_0b_1 - 4\ a_0a_1b_0 - 2a_2 - 12{a_0}^2a_1$
xy^2	$-12a_2$
x^2y^2	$-{b_1}^2 - 4a_1b_1 - 2a_2b_0 - 24a_0a_2 - 12{a_1}^2$
x^3y^2	$-a_0{b_1}^2 + (-a_1b_0 - 4a_0a_1)\ b_1 + \left(-2a_0a_2 - 4{a_1}^2\right)\ b_0 - 12{a_0}^2a_2 - 12a_0{a_1}^2$
xy^3	$-16a_1a_2$
x^2y^3	$-a_1{b_1}^2 + \left(-a_2b_0 - 4{a_1}^2\right)\ b_1 - 6a_1a_2b_0 - 16a_0a_1a_2 - 4{a_1}^3$
y^4	$-8{a_2}^2$
xy^4	$-a_2{b_1}^2 - 4a_1a_2b_1 - 2{a_2}^2b_0 - 8a_0{a_2}^2 - 4{a_1}^2a_2$

TABLE 2. Coefficients of b in $adx + bdy$.

generated by A, B, and C. All the ideals returned by the algorithm contain a_2, except the one that is generated only by x and y. Since we are assuming that $a_2 \neq 0$, we are only concerned with this last ideal. Therefore, when $a_2 \neq 0$, the only singularity of $\mathcal{G}$ is the one with $x = y = 0$. But the only point of $\mathbb{P}^2$ with $x = y = 0$ is $[0 : 0 : 1]$. Hence, this is the only singularity of $\mathcal{G}$ under the hypothesis that $a_2 \neq 0$. Since the 1-form $adx + bdy$ defined above has algebraic multiplicity 2 at $[0 : 0 : 1]$ and the homogeneous component of a is $4xy$, and that of b if $-4x^2$, it follows that $[0 : 0 : 1]$ is a dicritical singularity of $\mathcal{G}$. Finally, the line at infinity $z = 0$ cannot be an algebraic solution of $\mathcal{G}_0$, because neither the homogenization of a, nor that of b, is a multiple of z. □

Let A, B and C be homogeneous polynomials of degree $d + 1 \geq 2$ in x, y and z with complex coefficients. Since the vector space of homogeneous polynomials of degree $d+1$ in $\mathbb{C}[x, y, z]$ has dimension $\binom{d+2}{2}$, the triple (A, B, C) can be regarded as a point in a projective space of dimension $N_d = 3\binom{d+2}{2}-1$. Under this identification, every holomorphic foliation of degree $d + 1$ corresponds to a point of $\mathbb{P}^{N_d}$.

Theorem 3.2. *The projective closure $\mathfrak{G}$ of the family $\mathcal{G}(a_0, a_1, a_2, b_0, b_1)$ is an irreducible subset of $\mathbb{P}^{N_4}$.*

Proof. Let $\omega = adx + bdy$ be the 1-form that defines the foliation $\mathcal{G}(a_0, a_1, a_2, b_0, b_1)$ in the open set $z \neq 0$ and let A and B be the homogenizations of a and b with respect to z. The homogeneous 1-form that corresponds to ω is $\Omega = Adx + Bdy + Cdz$, where $xA + yB = -zC$. Denoting by α_{ij}, and β_{ij} the coefficients of $x^i y^j z^{5-i-j}$ in A and B, we have that $\gamma_{i,j} = -(\alpha_{i-1,j} + \beta_{i,j-1})$. In particular, all these coefficients are polynomials in $\mathbb{C}[a_0, a_1, a_2, b_0, b_1]$. Now, by Tables 1 and 2,

$$a_0 = -\frac{\beta_{3,0}}{12},\ a_1 = -\frac{\beta_{2,1}}{12},\ a_2 = \frac{\alpha_{0,3}}{4},\ b_0 = \frac{\alpha_{2,1} + \beta_{3,0}}{4} \text{ and } b_1 = \frac{\alpha_{1,2} + \beta_{2,1}}{4}.$$

Set $\mathbb{I} = \{\alpha_{0,3}, \alpha_{1,2}, \alpha_{2,1}, \beta_{2,1}, \beta_{3,0}\}$. Thus, for all $0 \leq i, j \leq 5$ such that $\alpha_{i,j}, \beta_{i,j} \notin \mathbb{I}$, there exist polynomials $\phi_{i,j}, \psi_{i,j} \in \mathbb{C}[\mathbb{I}]$, such that $\alpha_{i,j} = \phi_{i,j}$ and $\beta_{i,j} = \psi_{i,j}$. Hence, the ideal J generated by

$$\alpha_{i,j} - \phi_{i,j}, \beta_{i,j} - \psi_{i,j} \text{ and } \gamma_{i,j} + \phi_{i-1,j} + \psi_{i,j-1},$$

in the polynomial ring $A = \mathbb{C}[\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j} \mid 0 \leq i, j \leq 5]$ is prime, because it is defined by a system of linear polynomials in triangular form. Since $\alpha_{1,1} = 4$, the algebraic set defined by J is an affine variety in the open set $\alpha_{1,1} \neq 0$ of $\mathbb{P}^{N_4}$. Moreover, this set is irreducible, because J is prime. Therefore, so is its projective closure $\mathfrak{G}$ in $\mathbb{P}^{N_4}$. □

Let $\mathcal{S}_n$ and $\mathcal{H}_n$ be the sets of homogeneous polynomials, and homogeneous 2-forms, of degree n in x, y and z with complex coefficients and consider the projection

$$\pi_n : \mathfrak{G} \times \mathbb{P}(\mathcal{S}_n) \times \mathbb{P}(\mathcal{H}_4) \longrightarrow \mathfrak{G}.$$

The set

$$\mathcal{X}_n = \{(\Omega, F, \eta) \in \mathfrak{G} \times \mathcal{S}_n \times \mathcal{H}_4 \mid \Omega \wedge dF = \eta F\}$$

is closed in $\mathfrak{G} \times \mathbb{P}(\mathcal{S}_n) \times \mathbb{P}(\mathcal{H}_d)$. Since π_n is a proper map, the image $\pi_n(\mathcal{X}_n)$ is closed in $\mathfrak{G}$. Thus, either

$$\pi_n(\mathcal{X}_n) = \mathfrak{G} \text{ or } \dim(\pi_n(\mathcal{X}_n)) < \dim(\mathfrak{G}) \tag{3.1}$$

because $\mathfrak{G}$ is irreducible. See also [12, pp. 158–160].

Recall that a *very generic element* of an irreducible projective variety X has a property P if the set of points of X where P does not hold is contained in a countable union of hypersurfaces of X.

Theorem 3.3. *If $\mathcal{G}(a_0, a_1, a_2, b_0, b_1)$ has a foliation without any algebraic solutions, then a very generic element of $\mathfrak{G}$ does not have any algebraic solutions.*

Proof. Suppose that $\mathcal{G}(a_0, a_1, a_2, b_0, b_1)$ has a foliation without any algebraic solutions, Then, the corresponding point in $\mathfrak{G}$ is not contained in $\pi_n(\mathcal{X}_n)$ for any $n \geq 1$. Hence, by (3.1), $\dim(\pi_n(\mathcal{X}_n)) < \dim(\mathfrak{G})$ for all $n \geq 1$. In particular, all the $\pi_n(\mathcal{X}_n)$ are contained in hypersurfaces of $\mathfrak{G}$. □

4. The example and its properties

Taking

$$a_0 = b_0 = b_1 = a_1 = 0 \text{ and } a_2 = 1/2,$$

into the 1-form ω that defines $\mathcal{G}(a_0, a_1, a_2, b_0, b_1)$ and dividing the resulting 1-form by 4, we find the 1-form $\omega_0 = \widetilde{a}dx + \widetilde{b}dy$, where

$$\widetilde{a} = 2xy + 2y^3 + 2x^4 + x^3y^2 \text{ and } \widetilde{b} = -(2x^2 + 6xy^2 + 4y^4 + x^4y).$$

Let $\mathcal{G}_0$ be the foliation defined by the homogenization of ω_0. Then, $\Omega_0 = Adx + Bdy + Cdz$, where

$$\begin{aligned} A &= \left(2xyz^3 + 2y^3z^2 + 2x^4z + x^3y^2\right), \\ B &= \left(-2x^2z^3 - 6xy^2z^2 - 4y^4z - x^4y\right), \\ C &= \left(4xy^3z + 4y^5 - 2x^5\right). \end{aligned}$$

Since we chose $a_2 = 1/2$ when defining ω_0, it follows by Theorem 3.1, that $\mathcal{G}_0$ has only one singularity, which is dicritical. Moreover, the line at infinity $z = 0$ cannot be an algebraic solution of $\mathcal{G}_0$, because neither A, nor B, is a multiple of z.

4.1. **The isotropy group of $\mathcal{G}_0$.** Before we proceed to prove that $\mathcal{G}_0$ does not have any algebraic solutions, we must determine its isotropy group.

Proposition 4.1. *The isotropy group of $\mathcal{G}_0$ is generated by the projective transformation $\sigma(x, y, z) = (\zeta x, \zeta^3 y, z)$, where ζ is a primitive 5th root of unity. Moreover, $\sigma^*(\Omega_0) = \Omega_0$.*

Proof. Let σ be an isotropy of Ω_0. Since an isotropy permutes the singularities of the foliation, $p = [0 : 0 : 1]$ must be fixed under σ. Thus, if $M = (m_{i,j})_{1 \le i,j \le 3}$ is the matrix of σ, then $m_{1,3} = m_{2,3} = 0$. Moreover, since M represents an element of $\mathrm{PGL}_3(\mathbb{C})$, we can assume, without loss of generality, that $m_{3,3} = 1$, so that

$$\sigma(x, y, z) = (m_{11}x + m_{12}y, m_{21}x + m_{22}y, m_{31}x + m_{32}y + z). \tag{4.1}$$

A simple calculation shows that if $\sigma^*(\Omega_0) = \lambda\Omega_0$, then $\sigma^*(C) = \lambda C$, where C is the coefficient of dz in Ω_0. Equating the coefficients of xy^3z and y^4z on both sides of $\sigma^*(C) = \lambda C$, we get

$$\lambda = m_{22}^2(m_{11}m_{22} + 3m_{12}m_{21}) \text{ and } m_{12}m_{22}^3 = 0.$$

Since $\lambda \neq 0$, it follows from the first equation that $m_{22} \neq 0$ so that, from the second, $m_{12} = 0$, which, in turn, implies that $m_{11} \neq 0$. Doing the same with the coefficients of x^2y^2z, gives

$$0 = m_{21}m_{22}(m_{11}m_{22} + m_{21}m_{12}) = m_{11}m_{21}m_{22}^2,$$

where the last equality follows from $m_{12} = 0$. Thus, $m_{21} = 0$. So $\sigma(C) = \lambda C$ reduces to

$$-4m_{11}m_{22}^3xy^3(m_{31}x+m_{32}y+z)+2m_{11}^5x^5-4m_{22}^5y^5 = -4m_{11}m_{22}^3(-4xy^3z+2x^5-4y^5).$$

Hence, $m_{31} = m_{32} = 0$, $m_{11}^4 = m_{22}^3$ and $m_{11} = m_{22}^2$. It follows from the last two equations that both m_{11} and m_{22} are roots of $t^5 = 1$, so that $m_{11}^3 = m_{22}$. Therefore, if ζ is a primitive 5-th root of unit, then $\mathrm{iso}(\mathcal{G}_0)$ is generated by $\sigma(x, y, z) = (\zeta x, \zeta^3 y, z)$. Finally,

$$\lambda = m_{22}^3 m_{11} = (\zeta^3)^3\zeta = \zeta^{10} = 1,$$

so that $\sigma^*(\Omega_0) = \Omega_0$. $\square$

4.2. **Bounding the degree of algebraic solutions.** We begin by blowing up ω_0 at the origin. This gives rise to the foliation $\widetilde{\mathcal{G}}_0$ in the blowup $\mathcal{B}$ defined by

$$\omega_x = \big(-4x_1y_1^5 - 4y_1^3 + 2x_1\big)dx_1 + \big(-4x_1^2y_1^4 - 6x_1y_1^2 - x_1^3y_1 - 2\big)dy_1$$

in the open set $u \neq 0$ and by

$$\omega_y = \big(x_2^3y_2^3 + 2x_2^4y_2^2 + 2y_2 + 2x_2\big)dx_2 + \big(\big(2x_2^5 - 4\big)y_2 - 4x_2\big)dy_2$$

in the open set $v \neq 0$, see subsection 2.2 for the notation. This foliation has only the singularity $\widetilde{p_0}$, given by $x_2 = y_2 = 0$, which is a saddle-node, hence reduced. Moreover, the matrix (2.5) at this saddle-node is

$$\begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$$

whose eigenvalues are 0 and 6, with corresponding eigenvectors $(-1, 1)$ and $(2, 1)$. In particular, both the weak and strong separatrices of $\widetilde{\mathcal{G}}_0$ at $x_2 = y_2 = 0$ are transversal to the exceptional divisor $x_2 = 0$.

Let C be an algebraic solution of $\mathcal{G}_0$ and let $\mathcal{B}_C = \{C_1, \ldots, C_k\}$ be the set of irreducible branches of C in a small neighbourhood of p_0. As pointed out before, the number of irreducible branches of the curve C at p_0 need not coincides with its algebraic multiplicity at that same point. Let $q_i = \widetilde{C_i} \cap E$. If S and W are, respectively, the strong and the weak separatrices of $\widetilde{\mathcal{G}}_0$ at the saddle-node $\widetilde{p_0}$, then, by [3, p. 39–40],

$$\mathcal{Z}(\widetilde{\mathcal{G}}_0, \widetilde{C_i}, \widetilde{p_0}) = \begin{cases} 0 & \text{if } \widetilde{p_0} \notin \widetilde{C_i} \\ 1 & \text{if } \widetilde{C_i} = S \\ \mu + 1 & \text{if } \widetilde{C_i} = W, \end{cases} \tag{4.2}$$

where μ is the Milnor number of $\widetilde{\mathcal{G}_0}$ at $\widetilde{p_0}$. By (2.7),

$$\mathcal{Z}(\mathcal{G}_0, C, p_0) = \sum_{j=1}^{k} \mathcal{Z}(\mathcal{G}_0, C_j, p_0) - 2 \sum_{1 \le i < j \le k} (C_i \cdot C_j)_{p_0}. \tag{4.3}$$

We begin by computing the intersection multiplicities between branches at p_0, for which we need the following lemma.

Lemma 4.2. *If C_i and C_j are branches of C at p_0 then*

$$e_i = (\widetilde{C_i} \cdot E)_{q_i} = 1 \ \ \text{and} \ \ (C_i \cdot C_j)_{p_0} = \begin{cases} 1 & \text{if } \ \widetilde{C_i} \cap \widetilde{C_j} = \emptyset \\ 2 & \text{if } \ \widetilde{C_i} \cap \widetilde{C_j} = \{\widetilde{p_0}\}. \end{cases}.$$

Proof. Since the only point at which $\widetilde{\mathcal{G}}_0$ and E are not transversal is $\widetilde{p_0}$, it follows that $e_i = 1$, whenever $q_i \neq \widetilde{p_0}$. However, if $q_i = \widetilde{p_0}$, then $\widetilde{C_i}$ equals one of the two separatrices of ω_y at $\widetilde{p_0}$. But the separatrices are smooth and their tangents at $\widetilde{p_0}$ have the directions of $(-1, 1)$ and $(2, 1)$ in the coordinates (x_2, y_2). Therefore, $e_i = 1$ also in this case. Turning now to the intersection multiplicities between C_i and C_j, we have, by [5, Corollary 3.2.5, p. 73], that

$$\nu_{p_0}(C_i) = (\widetilde{C_i} \cdot E)_{q_i} = 1,$$

for all $1 \le i \le k$. Therefore, by [5, Lemma 3.3.4, p. 79]

$$(C_i \cdot C_j)_{p_0} = \nu_{p_0}(C_i)\nu_{p_0}(C_j) = 1.$$

when $\widetilde{C_i} \cap \widetilde{C_j} = \emptyset$ and

$$(C_i \cdot C_j)_{p_0} = \nu_{p_0}(C_i)\nu_{p_0}(C_j) + (\widetilde{C_i} \cdot \widetilde{C_j})_{\widetilde{p_0}} = 1 + 1 = 2$$

when $\widetilde{C_i} \cap \widetilde{C_j} = \{\widetilde{p_0}\}$. □

It follows from Lemma 4.2 that

$$\sum_{1 \leq i < j \leq k} (C_i \cdot C_j)_{p_0} = \frac{k(k-1)}{2} + \begin{cases} 1 & \text{if} \quad S, W \in \mathcal{B}_C \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

In the next lemma we compute the sum of GSV-indices in (4.3).

Lemma 4.3. *The following formula holds for the sum of the $\mathcal{Z}$-indices:*

$$\sum_{i=1}^{k} \mathcal{Z}(\mathcal{G}_0, C_i, p_0) = 2k + \begin{cases} 0 & \textit{if} \quad S, W \notin \mathcal{B}_C \\ 1 & \textit{if} \quad S \in \mathcal{B}_C \quad \textit{and} \quad W \notin \mathcal{B}_C \\ \mu + 1 & \textit{if} \quad W \in \mathcal{B}_C \quad \textit{and} \quad S \notin \mathcal{B}_C \\ \mu + 2 & \textit{if} \quad S, W \in \mathcal{B}_C \end{cases}$$

Proof. Using the notation of Lemma 4.2, we have, by [3, pp. 39], that

$$\mathcal{Z}(\mathcal{G}_0, C_i, p) = \mathcal{Z}(\widetilde{\mathcal{G}_0}, \widetilde{C_i}, q_i) - e_i^2 + \ell_0 e_i,$$

where ℓ_0 is the vanishing order of $\mathcal{G}_0$ along E, that is $\ell_0 = \nu_0(\mathcal{G}_0)$ in the non-dicritical case and $\ell_0 = \nu_0(\mathcal{G}_0) + 1$ in the dicritical case. Since p_0 is a dicritical singularity of $\mathcal{G}_0$ and $\nu_0(\mathcal{G}_0) = 2$, it follows that $\ell_0 = 3$. Now, $e_i = 1$, by Lemma 4.2, so that $\mathcal{Z}(\mathcal{G}_0, C_i, p) = \mathcal{Z}(\widetilde{\mathcal{G}_0}, \widetilde{C_i}, q_i) + 2$. Thus, by Proposition 2.2, equation (4.2), and Lemma 4.2,

$$\mathcal{Z}(\mathcal{G}_0, C_i, p) = 2 + \begin{cases} 0 & \text{if} \quad C_i \neq S, W \\ 1 & \text{if} \quad C_i = S \\ \mu + 1 & \text{if} \quad C_i = W, \end{cases}$$

from which the result of the lemma is a straightforward consequence. □

Combining Lemma 4.3 with equation (4.4), formula (4.3) becomes

$$\mathcal{Z}(\mathcal{G}_0, C, p_0) = 2k - k(k-1) + \gamma(C), \tag{4.5}$$

where

$$\gamma(C) = \begin{cases} 0 & \text{if} \quad S, W \notin \mathcal{B}_C \\ 1 & \text{if} \quad S \in \mathcal{B}_C \ \text{ and } \ W \notin \mathcal{B}_C \\ \mu + 1 & \text{if} \quad W \in \mathcal{B}_C \ \text{ and } \ S \notin \mathcal{B}_C \\ \mu & \text{if} \quad S, W \in \mathcal{B}_C \end{cases} \tag{4.6}$$

In order to use these formulas, we must first compute the Milnor number μ of $\widetilde{\mathcal{G}_0}$ at $\widetilde{p_0}$. This can be done directly, using Gröbner basis to compute the dimension of the vector space $\mathbb{C}[x, y]/I$, where I is the ideal generated by the coefficients of ω_0. However, since $\mathcal{G}_0$ has one singularity and degree 4, it follows by [4, Proposition 9.2, p. 176] that the Milnor number of $\mathcal{G}_0$ at its singularity is 21. Thus, by [4, Proposition 4.13, p. 88],

$$\mu = \mu(\widetilde{\mathcal{G}}_0, \widehat{p}) = 21 - 3^2 + 4 = 16. \tag{4.7}$$

γ	$27-4\gamma$	Pairs	k	Integer roots of (4.10)	Degree of curve
0	27	$(1,27)$, $(3,9)$	$8,3$	$-4,10,0,6$	$6,10$
1	23	$(1,23)$	7	$-3,9$	9
17	-41	$(-1,41)$	12	$-7,13$	13
16	-37	$(-1,37)$	11	$-6,12$	12

TABLE 3. Possible degrees for invariant curves.

Now, by [3, Proposition 3, p. 25], if C is an algebraic solution of $\mathcal{G}_0$, then

$$\mathcal{Z}(\mathcal{G}_0, C, p_0) = \mathcal{N}_{\mathcal{G}_0} \cdot C - C^2.$$

Since $\mathcal{G}_0$ has degree 4, it follows that $\mathcal{N}_{\mathcal{G}_0} = \mathcal{O}_{\mathbb{P}^2}(6)$, so that $Z(\mathcal{G}_0, C, p) = 6n - n^2$, where $n = \deg(C)$. Combining this with (4.5), we get

$$6n - n^2 = 2k - k(k-1) + \gamma(C), \tag{4.8}$$

where, by (4.6) and (4.7),

$$\gamma(C) = \begin{cases} 0 & \text{if } \; S, W \notin \mathcal{B}_C \\ 1 & \text{if } \; S \in \mathcal{B}_C \;\text{ and }\; W \notin \mathcal{B}_C \\ 17 & \text{if } \; W \in \mathcal{B}_C \;\text{ and }\; S \notin \mathcal{B}_C \\ 16 & \text{if } \; S, W \in \mathcal{B}_C \end{cases} \tag{4.9}$$

Proposition 4.4. *If C is an algebraic solution of $\mathcal{G}_0$ then* $\deg(C) \leq 13$.

Proof. By (4.8), $6n - n^2 = -k^2 + 3k + \gamma$, where $\gamma = \gamma(C)$. Therefore, if such a C exists, the quadratic equation

$$n^2 - 6n - k^2 + 3k + \gamma = 0, \tag{4.10}$$

has an integer solution. In particular, its discriminant must be a perfect square, so

$$36 + 4(k^2 - 3k - \gamma) = 4q^2,$$

for some integer q. Hence,

$$4q^2 - (2k-3)^2 = 27 - 4\gamma,$$

which is equivalent to

$$(2q - 2k + 3)(2q + 2k - 3) = 27 - 4\gamma.$$

We will say that (f_1, f_2) is a pair of factors for $27 - 4\gamma$ if $f_1 \leq f_2$ are integers such that $27 - 4\gamma = f_1 f_2$. For each such pair, we have a system

$$2q - 2k + 3 = f_1 \;\text{ and }\; 2q + 2k - 3 = f_2,$$

whose solution gives

$$k = \frac{f_2 - f_1 + 6}{4}.$$

Now, by (4.9), $27 - 4\gamma \in \{27, 23, -41, -37\}$. The factor pairs for these numbers, with the corresponding values of k and n are listed in Table 3. Since the difference between such factors is never even, it follows that k cannot be an integer in this case. The statement of the proposition is an immediate consequence of the table.

□

4.3. **Searching for algebraic solutions.** Now that we know an upper bound on the degree of the algebraic solutions of $\mathcal{G}_0$, we can search for them using a computer to apply the method of undetermined coefficients. Since, by Theorem 3.1, $z = 0$ cannot be an algebraic solution of $\mathcal{G}_0$, we need only search for algebraic solutions in $\mathbb{C}^2$.

Let $f \in \mathbb{C}[x, y]$ be an algebraic solution of $\mathcal{G}_0$ with co-factor $g \in \mathbb{C}[x, y]$. Then, $\deg(g) \leq \deg(\mathcal{G}_0) = 4$. Moreover, since $\sigma^*(\omega_0) = \omega_0$, by Proposition 4.1 , it follows from Lemma 2.1, that $\sigma^*(g) = \zeta^{-4}g = \zeta g$, where $\zeta^5 = 1$. In particular, $g_3 = 0$ and $g_2 = \alpha y^2$, for some complex number α. As we will see below, the other two homogeneous components of g can be given a more precise description.

Denoting by n the degree of f and by m its algebraic multiplicity at the origin,

$$f = \sum_{j=m}^{n} f_j,$$

where f_j is the homogeneous component of degree j of f. Since we are going to work with the homogeneous components of f, it is convenient to write the vector field dual to ω_0 as

$$D = (2x + x^3y + 2y^2)E + (4xy^2 + 4y^4)\frac{\partial}{\partial x} + 2x^4\frac{\partial}{\partial y},$$

where E is the Euler vector field, because then we can use the fact that $E(f_j) = jf_j$. As we saw in page 3, the equation $\omega_0 \wedge df = gf dx \wedge dy$ is equivalent to $D(f) = gf$. Equating homogeneous components of degree $j + 4$ on both sides of $D(f) = gf$, for $m - 3 \leq j \leq n$, we get

$$(4.11)\quad 2(j+3)xf_{j+3} + jx^3yf_j + 2(j+2)y^2f_{j+2} + 4xy^2\frac{\partial f_{j+2}}{\partial x} + 4y^4\frac{\partial f_{j+1}}{\partial x} + 2x^4\frac{\partial f_{j+1}}{\partial y} = g_1f_{j+3} + g_2f_{j+2} + g_4f_j,$$

The equations corresponding to $j = m-3$ and $j = n$ give $g_1 = 2mx$ and $g_4 = nx^3y$. Substituting the formulas for $g_1, \ldots, g_4$, found above, into (4.11) and rearranging the terms we get

$$(4.12)\quad 2(m-j-3)xf_{j+3} + (\alpha - 2j - 4)y^2f_{j+2} + (n-j)x^3yf_j - \left(4xy^2\frac{\partial f_{j+2}}{\partial x} + 4y^4\frac{\partial f_{j+1}}{\partial x} + 2x^4\frac{\partial f_{j+1}}{\partial y}\right) = 0.$$

For $m \leq i \leq n$, let

$$f_j = \sum_{\nu=0}^{j} c_{j,\nu}x^\nu y^{j-\nu}.$$

Thus, (4.12) can be rewritten as

$$2(m-j-3)\left(\sum_{\nu_1=0}^{j+3} c_{j+3,\nu_1}x^{\nu_1+1}y^{j+3-\nu_1}\right)+(\alpha-2j-4)\left(\sum_{\nu_2=0}^{j+2} c_{j+2,\nu_2}x^{\nu_2}y^{j+4-\nu_2}\right)$$
$$+(n-j)\left(\sum_{\nu_3=0}^{j} c_{j,\nu_3}x^{\nu_3+3}y^{j+1-\nu_3}\right)-4\left(\sum_{\nu_4=0}^{j+2}\nu_4 c_{j+2,\nu_4}x^{\nu_4+1}y^{j+4-\nu_4}\right)$$
$$-4\left(\sum_{\nu_5=1}^{j+1}\nu_5 c_{j+1,\nu_5}x^{\nu_5-1}y^{j+5-\nu_5}\right)-2\left(\sum_{\nu_6=0}^{j+1}(j+1-\nu_6)c_{j+1,\nu_6}x^{\nu_6+4}y^{j+1-\nu_6}\right)=0.$$

From now on we will assume that $c_{j,\nu}=0$ when $j<m$, $j>n$, or $m\le j\le n$ but ν is less than 0 or greater j. Collecting the coefficients of $x^r y^{j+4-r}$ under each summation sign we get

$$\begin{aligned}(4.13)\quad 2(m-j-3)c_{j+3,r-1}&-(2j+4r+4-\alpha)c_{j+2,r}\\ &-4(r+1)c_{j+1,r+1}-2(j+5-r)c_{j+1,r-4}+(n-j)c_{j,r-3}=0,\end{aligned}$$

where $m-1\le r\le n+3$. Note that these are quadratic equations because α is also a variable.

Theorem 4.5. *The foliation $\mathcal{G}_0$ has no algebraic solutions.*

Proof. As we have already pointed out, it follows from Theorem 3.1 that the line at infinity $z=0$ cannot be an algebraic solution of $\mathcal{G}_0$. Now, by Proposition 4.4, if $f\in\mathbb{C}[x,y]$ is an algebraic solution of $\mathcal{G}_0$, then $\deg(f)\le 13$. Moreover, $f(0,0)=0$ by [12, Proposition 4.1, p. 126]. By taking $m=1$ and $j=0$ when $n=1$, the system (4.13) reduces to

$$-4(r+1)c_{1,r+1}-2(5-r)c_{1,r-4}=0$$

from which we get $c_{1,0}=c_{1,1}=0$, because $c_{1,-5}=c_{1,-4}=0$. Therefore, $\mathcal{G}_0$ does not have any linear algebraic solutions. Similarly, to determine whether $\mathcal{G}_0$ has any algebraic solutions, we have to solve one quadratic system for each pair (m,n) with $1\le m<n\le 13$. Using Axiom's Gröbner Factorization Algorithm to do this, we find that the only solution in each case is $c_{j,\nu}=0$ for all choices of j and ν. □

Together with Theorem 3.3, this completes the proof of our main theorem.

Theorem 4.6. *A very generic elements of the projective closure $\mathfrak{G}$ of the family $\mathcal{G}(a_0,a_1,a_2,b_0,b_1)$ does not have any algebraic solutions.*

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