Catching Tangent Curves in Fields of Lines

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Abstract. In 1878 Gaston Darboux published a method for finding first integrals of differential systems with polynomial coefficients. Combined with tools from algebraic geometry and computer algebra, Darboux's method has opened up a vast field of work that is still being actively pursued. In this article, we introduce some of the key ideas and problems in this area.

1. INTRODUCTION. For the mathematicians of the 18th century, to solve an ordinary differential equation meant to express its solutions in analytic form. It is a testament to the fruitfulness of their methods that, to this day, our first courses on differential equations are still based on them. As Morris Kline points out in [**31**, p. 476] "[a]ll the elementary methods of solving first order equations were known by 1740." This includes separation of variables, which was discovered by James Bernoulli, and integrating factors, introduced independently by L. Euler and A. C. Clairaut. By the end of the 18th century, several equations of higher order had also been conquered. Among them are linear differential equations with constant coefficients, tackled by Euler in 1743.

Although, by the end of the 19th century, interest in the analytic solution of differential equations was waning, this did not stop G. Darboux from introducing a new method that would lead to a line of research that is still very much active today. Darboux's starting point was A. Clebsch's observation that "every differential equation of the first order establishes a relation between a point of the curve that satisfies the equation and its tangent at this point." [**14**, p. 60] This relation was stated using the language of projective geometry, a field that had been developing quickly since its introduction by Poncelet at the beginning of the century.

Although Darboux's paper did not immediately attract much attention, its importance was recognized by Poincaré, among others. As a result, the Académie des Sciences proposed, as the theme for its Grand Prix des Sciences Mathématiques for 1890, to "perfect in an important point the theory of differential equations of the first order and the first degree" [**23**, p. 1050]. The committee that analyzed the entries was composed by Hermite, Jordan, Poincaré, Darboux, and Picard, and the prize went to P. Painlevé, with L. Autonne receiving an honorable mention. The prize also had the effect of piquing Poincaré's interest in the area, in which he would eventually publish two important papers, [**35**] and [**36**], with exactly the same title. As we will see in Section 5, Poincaré begins the first of these papers with a problem that is still open, and that has played a key role in keeping interest alive in Darboux's ideas and methods throughout the second half of the 20th century and into our own days.

In the second paragraph of the same paper [**35**, p. 193], Poincaré refers to this work of Darboux as "l'oeuvre magistrale de M. Darboux." Similarly, Painléve in his 1895 Stockholm lectures on analytic differential equations, calls the same paper "une mémoire magistral." [**33**, p. 217] The high regard in which Darboux's paper was held is also shown by the fact that his method for finding first integrals made its way into such classic books as Ince's *Ordinary Differential Equations* [**26**, p. 19] and Jordan's *Cours d'Analyse* [**28**, pp. 27–37].

However, by the middle of the twentieth century, the study of ordinary differential equations had undergone a shift away from symbolic methods of solution, in part because it became clear that very few equations could be solved symbolically. Thus, applied mathematicians tended to solve equations using numerical methods, while pure mathematicians investigated the behavior of solutions using the qualitative methods that had also been pioneered by Poincaré.

The tide began to turn back with the advent of computer algebra in the 1970s and the realization that integration in elementary terms could be done algorithmically [**39**]. Thus, by 1983 M. J. Prelle and M. F. Singer had proposed an algorithm [**38**] to compute elementary first integrals of differential equations with polynomial coefficients that is heavily dependent on the ideas introduced by Darboux. However, although Prelle and Singer cite the papers of Poincaré and Painlevé mentioned above, they make no reference to Darboux at all. Indeed, most mathematicians who learned these techniques after 1979 did so through J.-P. Jouanolou's monograph [**29**], where Darboux's results are reworked in the language of modern algebraic geometry. Jouanolou's work also had a huge impact in the study of holomorphic foliations, which were eventually used in the development of algorithms for the symbolic solution of differential equations; see, for instance [**13, 19, 21**].

The aim of this article is to give an exposition of some of the results and techniques that have been developed in the wake of Darboux's "mémoire magistral." After describing the Darboux–Jouanolou method of finding rational first integrals of ordinary differential equations of the first order, we explain its relation to a problem proposed by Poincaré in [**35**, p. 193]. Although this is still an open problem, several partial solutions have been proposed, one of which is presented in Section 5.

Since Darboux's work was an early application of projective geometry to the study of differential equations, we begin with a brief review of the results of projective algebraic geometry that will be required throughout the paper.

2. PROJECTIVE ALGEBRAIC GEOMETRY.

The birth of projective geometry as a mathematical discipline can be traced to J.-V. Poncelet's *Traité des propriétés projectives des figures* [**37**], published in 1822. Although Poncelet had envisaged his methods as promoting a revival of the synthetic geometry of the ancients, as opposed to the analytical geometry of the moderns, mathematicians were quick to introduce coordinates in the study of projective spaces. Poncelet's *Traité* was mostly concerned with lines and conics, but by the time J. Plücker published his *System der analytischen Geometrie* in 1835, the use of coordinates had put algebraic curves of higher degree within reach of the methods of projective geometry. Since our interest is in algebraic curves, the short review of projective geometry provided in this section will be eminently analytic.

From this point of view, it is best to define the complex projective plane $\mathbb{P}^2$ as the quotient set of $\mathbb{C}^3 \setminus \{(0,0,0)\}$ by the equivalence relation that identifies any two points that belong to the same line through the origin. If $v = (x_0, y_0, z_0)$ is a nonzero complex vector, its equivalence class under this relation will be denoted by $[v] = [x_0 : y_0 : z_0]$; the numbers x_0, y_0 and z_0 are the *homogeneous coordinates* of the point $[v] \in \mathbb{P}^2$. Note that the homogeneous coordinates of a point are not unique. Indeed, if λ is a nonzero complex number, then

$$[v] = [x_0 : y_0 : z_0] = [\lambda x_0 : \lambda y_0 : \lambda z_0] = [\lambda v], \tag{1}$$

because v and λv define the same line through the origin of $\mathbb{C}^3$. The set L_∞ of points of $\mathbb{P}^2$ whose z-coordinate is zero is called the *line at infinity*.

Using homogeneous coordinates, a bijective map ε from $U = \mathbb{P}^2 \setminus L_\infty$ to $\mathbb{C}^2$ can be defined by the rule

$$\varepsilon([x_0 : y_0 : z_0]) = \left(\frac{x_0}{z_0}, \frac{y_0}{z_0}\right). \tag{2}$$

We can use ε^{-1} to map the points of an algebraic curve Γ of $\mathbb{C}^2$ into $U \subset \mathbb{P}^2$. But first, we need a definition. If $\phi \in \mathbb{C}[x, y]$ has total degree d, then its *homogenization* is the homogeneous polynomial of degree d in $\mathbb{C}[x, y, z]$ defined by

$$\Phi(x, y, z) = z^d \phi\left(\frac{x}{z}, \frac{y}{z}\right). \tag{3}$$

As a rule, we will use lowercase letters for polynomials in $\mathbb{C}[x, y]$ and the corresponding uppercase letters to denote their homogenizations. Assuming that Γ is the set of points where the nonconstant polynomial $\phi \in \mathbb{C}[x, y]$ vanishes, $\varepsilon^{-1}(\Gamma)$ is the set of points in $\mathbb{P}^2 \setminus L_\infty$ where Φ vanishes. However, it is more convenient to work with the *projectivization* $\overline{\Gamma} = \{[v] \in \mathbb{P}^2 \mid \Phi(v) = 0\}$ of Γ, which includes $\varepsilon^{-1}(\Gamma)$ and the points at the line at infinity where Φ vanishes.

There are several reasons why it is preferable to work with $\overline{\Gamma}$ rather than the curve Γ. The most obvious is that it allows mathematicians to justify the aphorism *parallel lines meet at infinity*. A deeper reason why the projectivization is to be preferred is that, unlike $\varepsilon^{-1}(\Gamma)$, the set $\overline{\Gamma}$ is both closed and compact with respect to the quotient topology of $\mathbb{P}^2$; see [**30**, pp. 34–40]

Actually, there is no reason to think of algebraic curves in $\mathbb{P}^2$ as projectivizations of curves in $\mathbb{C}^2$. Therefore, from now on, we take an *algebraic curve* of $\mathbb{P}^2$ to be a set of points of the form $\Gamma = \{[v] \in \mathbb{P}^2 \mid \Phi(v) = 0\}$, for some square-free homogeneous polynomial $\Phi \in \mathbb{C}[x, y, z]$. When Φ is irreducible, we say that Γ is *irreducible*. The *degree* of an algebraic curve Γ, which we denote by $\deg(\Gamma)$, is the degree of the square-free polynomial whose vanishing defines Γ. We finish this section with what is probably the best known result in the theory of algebraic curves.

Bézout's Theorem. *If Γ_1 and Γ_2 are distinct irreducible projective algebraic curves, then $\Gamma_1 \cap \Gamma_2$ is a nonempty set of at most $\deg(\Gamma_1)\deg(\Gamma_2)$ points.*

The theorem takes its name from E. Bézout, who published a proof in his *Théorie générale des équations algébriques*, see [**1**, p. xii]. However, the result had already been conjectured by C. Maclaurin and a proof had unsuccessfully been attempted by Euler; see [**15**, p. 6] for more details on the history of this result. Moreover, by our standards, Bézout's proof is far from satisfactory; a modern proof based on the same ideas already used by Bézout can be found in [**30**, pp. 51–62].

3. DARBOUX'S GEOMETRICAL APPROACH.

In Leibniz's approach to calculus, differential equations of the first order appeared as a tool to solve tangent problems; see [**24**, p. 242]. Such problems can be conveniently stated in terms of line fields, also known as direction fields. A *line field* is a map $\mathcal{F}$ that takes a point p of an open set of $\mathbb{C}^2$ to a line $\mathcal{F}(p)$ that contains p. The corresponding *tangent problem* can then be stated as: find a curve that is tangent to $\mathcal{F}$ at all of its points.

In particular, $\mathcal{F}$ is a *polynomial* line field if there exist polynomials $a, b \in \mathbb{C}[x, y, z]$ such that the line $\mathcal{F}(p)$, associated with a point $p = (u_0, v_0) \in \mathbb{C}^2$ by $\mathcal{F}$, has the form

$$a(p)(x - u_0) + b(p)(y - v_0) = a(p)x + b(p)y - c(p) = 0, \tag{4}$$

where $c = xa + yb$. Note that the same line field is defined by the equation obtained multiplying (4) by a nonzero complex number. This suggests that it is best to describe

$\mathcal{F}$ as the map from $\mathbb{C}^2$ to $\mathbb{P}^2$ given by $\mathcal{F}(p) = [a(p) : b(p) : -c(p))]$ with one caveat: a and b cannot both be zero at the same point, otherwise we do not have a well-defined line. Throughout this article all line fields will be polynomial.

In his 1878 paper, Darboux went one step further and defined a line field on the projective plane itself. Let $A, B, C \in \mathbb{C}[x, y, z]$ be homogeneous polynomials of degree $k \geq 1$ and set $\mathcal{F}([q]) = [A(q) : B(q) : C(q)]$. Since

$$[A(\lambda q) : B(\lambda q) : C(\lambda q)] = [\lambda^k A(q) : \lambda^k B(q) : \lambda^k C(q)] = [A(q) : B(q) : C(q)],$$

it follows that $\mathcal{F}([\lambda q]) = \mathcal{F}([q])$. Thus, $\mathcal{F}(p)$ is a well-defined map as long as A, *B, and C do not vanish simultaneously at p.* As in the previous paragraph, we will identify the point $[A(q) : B(q) : C(q)]$ with the line $A(q)x + B(q)y + C(q)z = 0$. Moreover, since we expect this line to pass through $q = [x_0 : y_0 : z_0]$, we also need to assume that $A(q)x_0 + B(q)y_0 + C(q)z_0 = 0$, for all $q \in \mathbb{P}^2$. This implies that $xA(x, y, z) + yB(x, y, z) + zC(x, y, z)$ is the zero polynomial.

Summing up, we have the following definition. A *line field* $\mathcal{F}$ of $\mathbb{P}^2$ is a map

$$\mathcal{F}([x : y : z]) = [A(x, y, z) : B(x, y, z) : C(x, y, z)], \tag{5}$$

where $A, B, C \in \mathbb{C}[x, y, z]$ are homogeneous polynomials of degree $k \geq 1$ that satisfy $xA + yB + zC = 0$. We say that $\mathcal{F}$ is *singular* at $q \in \mathbb{P}^2$ if $A(q) = B(q) = C(q) = 0$ and we denote by $\mathrm{Sing}(\mathcal{F})$ the set of all these points. Thus, the domain of $\mathcal{F}$ is $\mathbb{P}^2 \setminus \mathrm{Sing}(\mathcal{F})$. It follows from Bézout's theorem that $\mathrm{Sing}(\mathcal{F})$ is finite if $\gcd(A, B, C) = 1$. It is also true that all line fields of $\mathbb{P}^2$ have singular points; see [**42**, Proposition 10.2, p. 393].

Darboux realized that using the relation $xA + yB + zC = 0$ one can parameterize A, B, and C in terms of polynomials of degree $k - 1$. In order to do this we collect the terms that are divisible by x in B and C, writing $B = xN + R$, and $C = -xM + S$, with $R, S \in \mathbb{C}[y, z]$. Substituting these in $xA + yB + zC = 0$, we obtain

$$x(A + yN - zM) + yR + zS = 0.$$

But x does not divide $yR + zS$ unless $A = -yN + zM$ and $yR = -zS$. Thus, R must be a multiple of z. Writing $R = -zL$, we get $S = yL$, so that

$$A = -yN + zM, \quad B = xN - zL, \quad \text{and} \quad C = -xM + yL, \tag{6}$$

where L, M, and N are homogeneous polynomials of degree $k - 1$. The integer $k - 1$ is the *degree* of the line field $\mathcal{F}$.

This way of writing the entries of $\mathcal{F}$ is very helpful in the study of its *algebraic solutions*, which are the irreducible algebraic curves of $\mathbb{P}^2$ that are tangent to $\mathcal{F}$ at all of its points. Let Γ be such a curve and assume that it is defined by the vanishing of a nonconstant irreducible homogeneous polynomial $\Phi \in \mathbb{C}[x, y, z]$ of degree $\deg(\Phi) = m \geq 1$. Now $\mathcal{F}$ is tangent to Γ at a point $q \in \Gamma$ if $\nabla\Phi(q)$ and $\mathcal{F}(q)$ are collinear at q, which is equivalent to the vanishing of all the 2×2 minors of the matrix

$$\begin{bmatrix} A & B & C \\ \partial_x(\Phi) & \partial_y(\Phi) & \partial_z(\Phi) \end{bmatrix} = \begin{bmatrix} zM - yN & xN - zL & yL - xM \\ \partial_x(\Phi) & \partial_y(\Phi) & \partial_z(\Phi) \end{bmatrix}$$

for all $q \in \Gamma$. Adding $M + L + N$ times the Euler identity $x\partial_x(\Phi) + y\partial_y(\Phi) + z\partial_z(\Phi) = m\Phi$ to the sum

$$(B\partial_x(\Phi) - A\partial_y(\Phi)) + (C\partial_y(\Phi) - B\partial_z(\Phi)) + (A\partial_z(\Phi) - C\partial_x(\Phi)) = 0$$

of the three minors, we find that

$$-(x+y+z)(L\partial_x(\Phi)+M\partial_y(\Phi)+N\partial_z(\Phi)) = m(M+L+N)\Phi.$$

If $M+L+N$ is a multiple of $x+y+z$, then $D\mathcal{F}(\Phi) = G\Phi$, for some homogeneous polynomial G, where

$$D_{\mathcal{F}} = L\partial_x + M\partial_y + N\partial_z. \tag{7}$$

Otherwise, the fact that Φ is irreducible implies that, up to a constant multiple, $\Phi = x+y+z$. Then, from the vanishing of the minor $B\partial_x(\Phi) - A\partial_y(\Phi) = N(x+y+z) - z(M+L+N)$, we deduce that $x+y+z$ divides $M+L+N$. Thus, $D_{\mathcal{F}}(\Phi) = G\Phi$ also holds in this case. Summing up, we have the following result.

Proposition 1. *The algebraic curve Γ, defined by the vanishing of the nonconstant homogeneous irreducible polynomial $\Phi \in \mathbb{C}[x,y,z]$ is an algebraic solution of $\mathcal{F}$ if and only if $D_{\mathcal{F}}(\Phi) = G\Phi$ for some homogeneous polynomial $G \in \mathbb{C}[x,y,z]$.*

As usual we drop the subscript $\mathcal{F}$ and write simply D whenever the line field is clear from the context. The polynomial G in Proposition 1 is called the *cofactor* of Φ. Note that if Φ has degree d, then the degree of $D_{\mathcal{F}}(\Phi)$ is $\deg(\mathcal{F}) + d - 1$, while that of $G\Phi$ is $\deg(G) + d$, so $D_{\mathcal{F}}(\Phi) = G\Phi$ implies that

$$\deg(G) = \deg(\mathcal{F}) - 1. \tag{8}$$

Before we move on, we will determine the lines of $\mathbb{P}^2$ that are algebraic solutions of a line field of degree one, an example that was originally studied by C. G. J. Jacobi in [**27**]; see also [**14**, p. 70] and [**29**, pp. 8–19]. In this case L, M, N, as well as the polynomial F that defines the algebraic solution, are all linear; so we can write

$$[L \quad M \quad N] = \underbrace{[x \quad y \quad z]}_{u} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\mathcal{A}} \quad \text{and} \quad \Phi = \underbrace{[x \quad y \quad z]}_{u} \underbrace{\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}}_{v}, \tag{9}$$

where $\mathcal{A}$ and v are matrices with coefficients in $\mathbb{C}$. Now it follows from (8) that $G = \lambda \in \mathbb{C}$ when $\deg(\mathcal{F}) = 1$. Thus, in this case, $D_{\mathcal{F}}(\Phi) = G\Phi$ is equivalent to $\mathcal{A}v = \lambda v$. Therefore, the line of $\mathbb{P}^2$ defined by the polynomial $\Phi = uv$ is an algebraic solution of $\mathcal{F}$ if and only if v is an eigenvector of $\mathcal{A}$. In particular, *line field of degree one has at least one and at most three algebraic solutions of degree one*. For example, the line field

$$\mathcal{F}_0(x,y,z) = [2z - 4y + 14x \,:\, 8z + 11y + 2x \,:\, 11z + 5y - 4x]$$

has the two solutions $f_1 = 2z - y + 2x$ and $f_2 = 2z + 2y - x$. Figure 1 illustrates this line field and its linear solutions at the points with real coordinates in the set $U = \{[x:y:z] \in \mathbb{P}^2 \mid z \neq 0\}$. The line field is represented by the direction vectors of the corresponding lines.

However, Jouanolou showed in [**29**, Théoreme 1.1, p. 158] that, in strong contrast to this, most line fields of $\mathbb{P}^2$ of degree greater than one have no algebraic solution. We end with a result that will be used at the beginning of the next section.

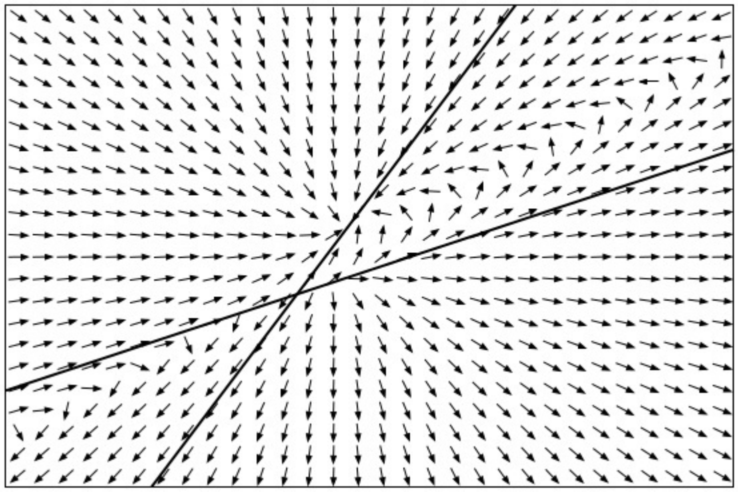

Figure 1. The line field $\mathcal{F}_0$ and its linear algebraic solutions.

Lemma 2. *Let $\mathcal{F}$ be a line field and let $\Phi, G \in \mathbb{C}[x, y, z]$ be homogeneous polynomials. If $D_{\mathcal{F}}(\Phi) = G\Phi$, then each irreducible factor of Φ defines an algebraic solution of $\mathcal{F}$.*

Proof. If P is an irreducible factor of Φ, then we can write $\Phi = P^e Q$, where $e > 0$ is an integer and $Q \in \mathbb{C}[x, y, z]$ is not divisible by P. Then

$$D_{\mathcal{F}}(\Phi) = eP^{e-1}QD_{\mathcal{F}}(P) + P^e D_{\mathcal{F}}(Q).$$

Substituting this into $D_{\mathcal{F}}(\Phi) = G\Phi$, and cancelling P^{e-1} throughout the equation, we get

$$eQD_{\mathcal{F}}(P) + PD_{\mathcal{F}}(Q) = GPQ.$$

Therefore, since $\gcd(P, Q) = 1$ by hypothesis, it follows that P divides $D_{\mathcal{F}}(P)$. Hence, P is an algebraic solution of $\mathcal{F}$. ■

4. DARBOUX'S METHOD. Throughout this section $\mathcal{F}$ will be the line field of $\mathbb{P}^2$ defined by (5) and L, M, and N will be the homogeneous polynomials of degree $k \geq 2$ defined in (6). We saw in the last section that the curve given by the vanishing of a nonconstant irreducible homogeneous polynomial $\Phi \in \mathbb{C}[x, y, z]$ is an algebraic solution of $\mathcal{F}$ if there exists a polynomial $G \in \mathbb{C}[x, y, z]$ such that $D(\Phi) = G\Phi$, with $D = D_{\mathcal{F}}$ the linear operator defined in (7). An important special case occurs when $\mathcal{F}$ has two algebraic solutions, defined by nonconstant homogeneous polynomials Φ_1 and Φ_2, both of them of the same degree d and with the same cofactor G. Indeed, in this case

$$D\left(\frac{\Phi_1}{\Phi_2}\right) = \frac{\Phi_2 D(\Phi_1) - \Phi_1 D(\Phi_2)}{\Phi_2^2} = \frac{\Phi_2 \cdot \Phi_1 G - \Phi_1 \cdot \Phi_2 G}{\Phi_2^2} = 0. \qquad (10)$$

Borrowing the terminology from differential equations, we will say that the rational function Φ_1/Φ_2 is a *first integral* of $\mathcal{F}$. The existence of a first integral implies that there is an algebraic solution of $\mathcal{F}$ through every point of $\mathbb{P}^2$. To see this, let $p \in \mathbb{P}^2$. If $\Phi_2(p) = 0$, for some $p \in \mathbb{P}^2$, then the algebraic solution $\Phi_2 = 0$ contains p; otherwise, taking $c = \Phi_1(p)/\Phi_2(p)$, we have that

$$D(\Phi_1 - c\Phi_2) = D(\Phi_1) - cD(\Phi_2) = G\Phi_1 - cG\Phi_2 = G(\Phi_1 - c\Phi_2).$$

Therefore, by Lemma 2, there is an irreducible factor of $\Phi_1 - c\Phi_2$ that defines an algebraic solution of $\mathcal{F}$ through p.

Darboux realized that this argument could be generalized to produce multivalued first integrals of $\mathcal{F}$. However, to keep within the algebraic setting of this article we will prove instead a result of Jouanolou [**29**, Théoréme 3.3, p. 102], who used Darboux's strategy to characterize line fields with rational first integrals.

Lemma 3. *Let V be a complex vector space of dimension $m \geq 1$ and let $v_1, \dots, v_{m+4}$ be vectors in V. Reordering the v's, if necessary, there exist, for $i = 1, 2$, complex numbers $\alpha_{i,3}, \dots, \alpha_{i,m+4} \in \mathbb{C}$ such that*

$$v_i + \sum_{j=3}^{m+4} \alpha_{i,j} v_j = 0 \quad \textit{and} \quad 1 + \sum_{j=3}^{m+4} \alpha_{i,j} = 0. \tag{11}$$

Proof. Reorder the v's so that $v_k, \dots, v_{m+4}$ is a basis of the subspace W of V generated by $v_1, \dots, v_{m+4}$ and consider the map $\theta : \mathbb{C}^{m+7-k} \to W \times \mathbb{C}$ defined by

$$\theta(\beta_1, \beta_3, \beta_k \dots, \beta_{m+4}) = \left(\beta_1 v_1 + \beta_3 v_3 + \sum_{j=k}^{m+4} \beta_j v_j,\ \beta_1 + \beta_3 + \sum_{j=k}^{m+4} \beta_j \right).$$

Since $\dim(W) = m + 5 - k$, it follows from the rank-nullity theorem that the kernel of θ is nonzero. Thus, there exist $\beta_1, \beta_3, \beta_k, \dots, \beta_{m+4} \in \mathbb{C}$ such that

$$\beta_1 v_1 + \beta_3 v_3 + \sum_{j=k}^{m+4} \beta_j v_j = \beta_1 + \beta_3 + \sum_{j=k}^{m+4} \beta_j = 0.$$

Moreover, β_1 or β_3 must be nonzero because $v_k, \dots, v_{m+4}$ is a basis of W. Thus, swapping the first two vectors, if necessary, we can assume that $\beta_1 \neq 0$. Setting

$$\alpha_{1,2} = \alpha_{1,4} = \cdots = \alpha_{1,k-1} = 0 \text{ and } \alpha_{1,j} = \frac{\beta_j}{\beta_1} \text{ for } j = 3, 5, k, \dots, d+4,$$

we get (11) when $i = 1$. The corresponding equations when $i = 2$ follow by the same argument with v_1 and v_3 replaced, respectively, by v_2 and v_4. ■

Before stating the theorem, we establish a formula that will be used twice in its proof. Let $E = x\partial_x + y\partial_y + z\partial_z$ be the Euler operator and let ψ be a function of the variables x, y, and z that is holomorphic on an open set of $\mathbb{C}^3$. Then, by (6),

$$zD(\psi) - NE(\psi) = A\partial_y(\psi) - B\partial_x(\psi). \tag{12}$$

Theorem 4. *If a line field $\mathcal{F}$ of degree k has at least $4 + k(k+1)/2$ algebraic solutions, then it has a rational first integral.*

Proof. Let $m = k(k+1)/2$ and let $\Phi_1, \ldots, \Phi_{m+4}$ be nonconstant homogeneous polynomials in $\mathbb{C}[x, y, z]$ that define algebraic solutions of $\mathcal{F}$. We will denote the corresponding cofactors by $G_1, \ldots, G_{m+4}$. Since $\mathcal{F}$ has degree k, it follows from (8) that $G_1, \ldots, G_{m+4}$ belong to the m-dimensional complex vector space of homogeneous polynomials of degree $k-1$. Thus, by Lemma 3 it is possible to order the G's so that, for some choice of $\alpha_{i,3}, \ldots, \alpha_{i,m+4} \in \mathbb{C}$,

$$G_i + \sum_{j=3}^{m+4} \alpha_{ij} G_j = 1 + \sum_{j=3}^{m+4} \alpha_{ij} = 0 \quad \text{for} \quad i = 1, 2. \tag{13}$$

Set

$$\psi_i = \log(\Phi_i) + \sum_{j=3}^{m+4} \alpha_{ij} \log(\Phi_j), \quad \text{for} \quad i = 1, 2.$$

Differentiating ψ_i with respect to a variable $\xi \in \{x, y, z\}$, we get

$$\partial_\xi(\psi_i) = \frac{\partial_\xi(\Phi_i)}{\Phi_i} + \sum_{j=3}^{m+4} \alpha_{ij} \frac{\partial_\xi(\Phi_j)}{\Phi_j}. \tag{14}$$

Hence, taking $D = D_{\mathcal{F}}$ as in (7), we have

$$D(\psi_i) = \frac{D(\Phi_i)}{\Phi_i} + \sum_{j=3}^{m+4} \alpha_{ij} \frac{D(\Phi_j)}{\Phi_j}.$$

Taking into account that $D(\Phi_j) = G_j \Phi_j$, for $1 \leq j \leq m+4$, we conclude that

$$D(\psi_i) = G_i + \sum_{j=3}^{m+4} \alpha_{ij} G_j = 0.$$

A similar argument, this time using the second equality of (13), shows that $E(\psi_i) = 0$. Thus, by (12), $0 = -B\partial_x(\psi_i) + A\partial_y(\psi_i)$. Therefore,

$$\rho_i = \frac{\partial_x(\psi_i)}{A} = \frac{\partial_y(\psi_i)}{B} \tag{15}$$

is a rational function. Since there are at least four nonconstant, irreducible Φ's, the derivatives $\partial_x(\psi_i)$ and $\partial_y(\psi_i)$ cannot both be zero. Moreover,

$$\deg(\partial_x(\Phi_j)) - \deg(\Phi_j) - \deg(A) = -1 - (k+1) = -k - 2, \tag{16}$$

for all $1 \leq j \leq m+4$. Now, by (6),

$$\partial_y(A) - \partial_x(B) = -2N - E(N) + z\,\mathrm{div}(D) = -(k+2)N + z\,\mathrm{div}(D),$$

where $\mathrm{div}(D) = \partial_x(L) + \partial_y(M) + \partial_z(N)$. But, by (15),

$$0 = \partial_y(\partial_x(\psi_i)) - \partial_x(\partial_y(\psi_i)) = A\partial_y(\rho_i) - B\partial_x(\rho_i) + \rho_i(\partial_y(A) - \partial_x(B)),$$

so that

$$A\partial_y(\rho_i) - B\partial_x(\rho_i) = -\rho_i(\partial_y(A) - \partial_x(B)) = ((k+2)N - z\,\mathrm{div}(D))\rho_i.$$

Therefore, by (12),

$$zD(\rho_i) - NE(\rho_i) = \rho_i((k+2)N - z\,\mathrm{div}(D)). \tag{17}$$

But, from (16) and the fact that numerator and denominator of ρ_i are homogeneous, we get that $E(\rho_i) = -(k+2)\rho_i$. Taking this into (17) and cancelling common terms, we deduce that $D(\rho_i) = -\rho_i\,\mathrm{div}(D)$, for $i = 1, 2$. Therefore, $D(\rho_1/\rho_2) = 0$ by an argument analogous to that used in (10). Moreover, using (14),

$$\frac{\rho_1}{\rho_2} = \frac{\partial_x(\psi_1)}{\partial_x(\psi_2)} = \frac{\partial_y(\psi_1)}{\partial_y(\psi_2)}$$

can be written as a quotient of homogeneous polynomials of the same degree. Finally, this rational function cannot be constant because $\partial_\xi(\psi_1)$ and $\partial_\xi(\psi_2)$ have distinct denominators. ■

The lower bound in Jouanolou's original result was $2 + k(k+1)/2$; we have increased it to $4 + k(k+1)/2$ in order to simplify the proof. It turns out that this does not weaken the result, because combining Theorem 4 with the argument preceding it, we have the following corollary; see [**29**, Théoréme 3.3, p. 102].

Corollary 5. *A line field has a rational first integral if and only if it has infinitely many algebraic solutions.*

The proof given above is adapted from [**9**, pp.13–14] and [**10**, p. 81]. The theorem was generalized by E. Ghys in [**22**]; see also [**20**, Theorem 2.1, p. 731], where its relation to model theory is also discussed.

5. THE POINCARÉ PROBLEM. Drawing inspiration from Darboux's work, and from later work by P. Painlevé and L. Autonne, in 1891 Poincaré published a paper [**35**] devoted to the study of polynomial differential equations, which begins as follows:

> In order to determine if a differential equation of the first order and the first degree is algebraically integrable, it is evidently enough to find an upper bound for the degree of the integral; after that one needs only to perform purely algebraic computations.

In Poincaré's terminology, a differential equation is *algebraically integrable* if it admits a rational first integral whose numerator and denominator are non-constant polynomials of the same degree. The *degree* of a rational first integral is the degree of its numerator and denominator. However, as Poincaré points out in the same paper, if ρ is a first integral of a given differential equation then

> one will get another form of the general integral on equating to a constant any polynomial with respect to [ρ]. As a result one cannot find an upper limit of the degree of the general algebraic integral, unless one finds some means of expressing, in the inequalities, that this integral is [indecomposable;]

to which he adds that, finding these *means*, is precisely what he aims to do in his paper. What Poincaré is getting at is that, if ρ is a rational first integral of a given differential equation, and $\psi \in \mathbb{C}[x, y, z]$, then it follows from the chain rule that $\psi \circ \rho$ is also a rational first integral of the same differential equation. Thus, one can only find the desired upper bound on the degree of ρ if one includes the hypothesis that ρ cannot be written as a composition of a polynomial with another rational first integral, which is what is meant by an indecomposable first integral.

But we saw in Section 4 that if Φ_1 and Φ_2 are polynomials such that Φ_1/Φ_2 is a rational first integral of a line field $\mathcal{F}$, then the irreducible factors of $\Phi_1 - c\Phi_2$ are algebraic solutions of $\mathcal{F}$, for all $c \in \mathbb{C}$. Therefore the first step in settling the question posed by Poincaré is to solve the following problem, which became known, in the 20th century, as *Poincaré's problem.*

Poincaré's Problem. *Given a line field of $\mathbb{P}^2$, find an upper bound on the degree of the algebraic solutions of $\mathcal{F}$.*

A partial solution to Poincaré's problem was proposed by Jouanolou, who showed in [**29**, Proposition 4.1, p. 126] that the degree of a *smooth* algebraic solution of $\mathcal{F}$ must be at most equal to $\deg(\mathcal{F}) + 1$. Corollary 5, which appears in the same monograph, has had a significant impact in the history of Poincaré's problem. Indeed, since a line field without a rational first integral has finitely many algebraic solutions, there must be an upper bound on the degree of these solutions. This shifted the focus from line fields with rational first integrals to those that do *not* have such integrals.

The first significant advance towards a solution of Poincaré's problem, after Jouanolou's work, was the article [**6**, Theorem 1, p. 891]. In it, D. Cerveau and A. Lins Neto showed that a nodal curve can only be an algebraic solution of $\mathcal{F}$ if its degree is at most $\deg(\mathcal{F}) + 1$. From the algorithmic point of view, this result has the advantage that there are hypotheses on $\mathcal{F}$ that force its algebraic solutions to be nodal curves [**12**, Proposition 2.3, p. 607]. A far more general bound, whose hypotheses can also be checked directly on the line field, was found by M. Carnicer in [**5**].

All these bounds require $\mathcal{F}$ to satisfy some extra hypothesis. Indeed, there cannot be an upper bound on the degree of an algebraic solution that holds for all line fields of a given degree. For example, the curve defined by the vanishing of $\Phi = z^{m-1}y - x^m$, which is a polynomial of degree m, is an algebraic solution of the line field $\mathcal{F}(x, y, z) = [myz : -xz : (1-m)xy]$. Indeed, writing the entries of $\mathcal{F}$ as in (6), we find that $L = x$, $M = my$, and $N = 0$, so that

$$L\partial_x(\Phi) + M\partial_y(\Phi) + N\partial_z(\Phi) = x(-mx^{m-1}) + my(z^{m-1}) = m\Phi.$$

Thus, (7) is satisfied with $G = m$. However, $\mathcal{F}$ has degree one. A far more refined example, of the same nature, was found by Lins Neto. Poincaré's attack on the problem in [**35, 36**], consisted in trying to find an upper bound that would work for all line fields of a given degree, with nondegenerate singularities of a fixed local analytic type. In [**32**, Main Theorem, p. 234] Lins Neto shows that this is not possible, by constructing line fields that satisfy all these properties, but whose rational first integrals are defined by polynomials of arbitrarily large degree.

We end this section with a watered-down version of S. Walcher's (partial) solution of the Poincaré problem; see [**41**, Theorem 3.4, p. 66]. Although this solution came after those in [**6, 5**], it is easier to explain without introducing a lot of machinery. But before we state it, we need a definition. The line at infinity L_∞ is said to be *transversal* to an algebraic curve Γ if Γ has a well-defined tangent line different from L_∞ at every point of $\Gamma \cap L_\infty$. When Γ is defined by the vanishing of a polynomial $\Phi \in \mathbb{C}[x, y, z]$, the transversality condition amounts to saying that, at every $p \in \Gamma \cap L_\infty$, one has $\partial_x(\Phi)(p) \neq 0$ or $\partial_y(\Phi)(p) \neq 0$.

Theorem 6 (S. Walcher [41]). *Let $\mathcal{F}$ be a line field of $\mathbb{P}^2$ that has the line at infinity L_∞ as one of its algebraic solutions. If an algebraic solution $\Gamma \neq L_\infty$ of $\mathcal{F}$ is transversal to L_∞, then* $\deg(\Gamma) \leq \deg(\mathcal{F}) + 1$.

Proof. If $\Phi(x, y, z)$ is a homogeneous polynomial of degree d that defines the curve Γ, then

$$\Gamma \cap L_\infty = \{[x_0 : y_0 : 0] \in \mathbb{P}^2 \mid \Phi(x_0, y_0, 0) = 0\}.$$

Writing Φ as a polynomial in z with coefficients in $\mathbb{C}[x, y]$, we have

$$\Phi(x, y, z) = \sum_{i=0}^{d} \Phi_i(x, y) z^{d-i}, \tag{18}$$

where Φ_i is a homogeneous polynomial of degree i in $\mathbb{C}[x, y]$. Hence,

$$0 = \Phi(x_0, y_0, 0) = \Phi_d(x_0, y_0).$$

Since $\Phi_d(x, y)$ is a homogeneous polynomial in two variables, it can be factored in the form

$$\Phi_d(x, y) = \prod_{i=1}^{d} (\alpha_i y - \beta_i x), \tag{19}$$

which implies that

$$\Gamma \cap L_\infty = \{[\alpha_i : \beta_i : 0] \mid 1 \leq i \leq d\}.$$

In particular, $\#(\Gamma \cap L_\infty) \leq d$. Now we bring in the transversality hypothesis which, as we have seen, is equivalent to saying that for every $1 \leq i \leq d$, one has $\partial_x \Phi_d(p_i) \neq 0$ or $\partial_y \Phi_d(p_i) \neq 0$. However, this holds if and only if $\alpha_i y - \beta_i x$ has multiplicity one in the factorization (19). Hence, the p_i's are all distinct, so that

$$\#(\Gamma \cap L_\infty) = \deg(\Gamma). \tag{20}$$

Assume now that p_i is a singular point of $\mathcal{F}$, for some $1 \leq i \leq d$, and note that the definition of transversality requires that $\nabla\Phi(p_i) \neq 0$, for all $1 \leq i \leq d$. Since Φ and L_∞ are both algebraic solutions of $\mathcal{F}$, it follows that $\nabla\Phi(p_i)$ and the tangent vector $(0, 0, 1)$ of L_∞ must be collinear with $\mathcal{F}(p_i)$, which contradicts the transversality assumption. Therefore, all the points of $\Gamma \cap L_\infty$ must be singularities of $\mathcal{F}$. Combining this with (20), we conclude that

$$\deg(\Gamma) = \#(\Gamma \cap L_\infty) \leq \#(\mathrm{Sing}(\mathcal{F}) \cap L_\infty). \tag{21}$$

Finally, we must count the number of singular points of $\mathcal{F}$ that belong to L_∞. Since we are assuming that L_∞ is an algebraic solution of $\mathcal{F}$, it follows from Proposition 1 and equation (7) that $D_{\mathcal{F}}(z) = N$ must be a multiple of z; say $N = z\widehat{N}$. Thus, by equation (6),

$$A = z(-y\widehat{N} + M) \quad \text{and} \quad B = z(x\widehat{N} - L),$$

In particular, both A and B are multiples of z. But if $p = [x_0 : y_0 : 0] \in \text{Sing}(\mathcal{F}) \cap L_\infty$, then

$$-x_0 M(x_0, y_0, 0) + y_0 L(x_0, y_0, 0) = C(p) = 0.$$

In other words, $x_0 y - y_0 x$ is a linear factor of $\Delta = xM(x, y, 0) + yL(x, y, 0)$. Moreover, Δ cannot be identically zero for, otherwise, C would also be a multiple of z and $\gcd(A, B, C)$ would not be 1. But Δ has as many distinct linear factors as its degree, which is equal to $\deg(\mathcal{F}) + 1$. Therefore, by (21),

$$\deg(\Gamma) \leq \#(\text{Sing}(\mathcal{F}) \cap L_\infty) \leq \deg(\mathcal{F}) + 1,$$

as we wished to prove. ■

The above result is a special case of the theorem proved by S. Walcher in [**41**, Theorem 3.4, p. 66]. Unlike Theorem 6, Walcher's result is effective: a condition is imposed on the field $\mathcal{F}$ that can be easily checked by a computer and that forces the curve Γ and the line L_∞ to be transversal; see [**12**, Proposition 2.3, p. 607].

6. CODA. The story that began with Darboux's 1878 paper is far from over. Two areas, in particular, have seen a number of advances in the last 20 years: the Poincaré problem and the effective calculation of Darboux polynomials. Ironically the latter corresponds to the "purely algebraic computations" that, according to Poincaré, one "need only perform" in order to get the algebraic solutions. Indeed, using the method of undetermined coefficients to find an algebraic solution of degree d, one gets a system of polynomial equations of degree two in $(d + 1)(d + 2)/2$ variables. Unfortunately, solving such a system quickly gets beyond the reach of most computers. However, several faster methods, based on completely different ideas have been proposed recently. Using these methods G. Chèze proved in [**8**] that the problem can be solved in polynomial time. Faster algorithms have been proposed, for example, in [**2, 7**].

There have also been significant advances in the solution of Poincaré's problem; both when the equations have a rational first integral, [**11, 19, 21**], and when there are only finitely many algebraic solutions [**13**]. Moreover, similar problems have been investigated in many other contexts, among them: projective spaces of dimension higher than two [**16, 40**]; varieties other than projective spaces [**3, 17**]; Pfaff equations [**3, 18**]; and line fields over fields of positive characteristic [**34**]. Several of these results have made it into books on the analytic theory of differential equations, such as [**4, 25**]. However, despite all these advances, the problem remains tantalizingly open, even for line fields of $\mathbb{P}^2$.

ACKNOWLEDGMENTS. This work was partially supported by CNPq grant 302585/2013-3. I have also benefited from the access to online journals provided by CAPES. I would also like to thank the referees for numerous suggestions that greatly improved this paper.

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